Session 3

Solving Linear and Quadratic Equations

and Absolute Value Equations

Solving Equations

An **equation** is a statement expressing the equality of two mathematical expressions. It may have numeric and variable terms on the left hand side (LHS) and similar terms on the right hand side (RHS):

$$\text{LHS} = \text{RHS}$$

$$3x + 2 = 27x - 4$$

We want to **solve** the equation, which means we want to get the variable by itself on the left side and the numeric values by themselves on the right side, if possible.

Solving Equations

Solving an equation means finding the value(s) the variable can take on to make the equation a true statement. Sometimes there are no such values:

$$x = x + 1$$

Sometimes there are <u>multiple</u> solutions:

$$x^2 = 4$$

This equation has <u>two</u> solutions: 2 and -2.

Solving Linear Equations

To solve linear equations, we can use the additive and multiplicative properties of equality.

The additive property of equality:

If $a = b$, then $a + c = b + c$.

If we choose c to be the additive inverse of a term, we can add or subtract it from both sides of the equation, and take steps to isolate the variable term.

$$\begin{aligned} 3 + x &= 5 \\ 3 - 3 + x &= 5 - 3 \\ x &= 2 \end{aligned}$$

Solving Linear Equations

The multiplicative property of equality:

If $a = b$, then $ac = bc$

If we choose c to be the multiplicative inverse of a term, we can multiply both sides of the equation by the multiplicative inverse and “get rid of” or ”divide out” a coefficient on the variable we are trying to isolate.

$$\begin{aligned} \left(\frac{3}{5}\right)x &= 10 \\ \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)x &= \left(\frac{5}{3}\right)10 \\ x &= \frac{50}{3} \end{aligned}$$

Steps in Solving Linear Equations

1. Remove all parentheses by using the distributive property.
2. Use the additive property of equality to move all variable terms to the LHS and all constant terms to the RHS.
3. Simplify by combining like terms.
4. Use the multiplicative property of equality to change the coefficient of the variable to 1.
5. Simplify the RHS. When the variable is alone on the LHS, the RHS is the solution to the equation.
6. Check your answer by plugging the solution back into the original equation. (Always!)

Example: Solving Linear Equations

$$\begin{aligned}
3(x-2)+3 &= 2(6-x) & \\
3x-6+3 &= 12-2x & \text{Distributive property} \\
3x-3 &= 12-2x & \text{Combine like terms} \\
3x+2x-3 &= 12-2x+2x & \text{Add } 2x \text{ to each side} \\
5x-3 &= 12 & \text{Combine like terms} \\
5x-3+3 &= 12+3 & \text{Add 3 to each side} \\
5x &= 15 & \text{Combine like terms} \\
\left(\tfrac{1}{5}\right)5x &= \left(\tfrac{1}{5}\right)15 & \text{Multiply each side by } \tfrac{1}{5} \\
x &= 3 & \text{This is the solution} \\
\text{3(3-2) + 3} &= \text{2(6-3)?} & \text{Check your answer} \\
\text{Yes!} & &
\end{aligned}$$

Finding Linear Equations

If you know two points, (x_1, y_1) (the x and y coordinates of the first point) and (x_2, y_2) (the x and y coordinates of the second point, you can find the equation for a line:

1. The slope is the change in y divided by the change in x:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

2. Set up a formula in the standard form, using the slope you calculated

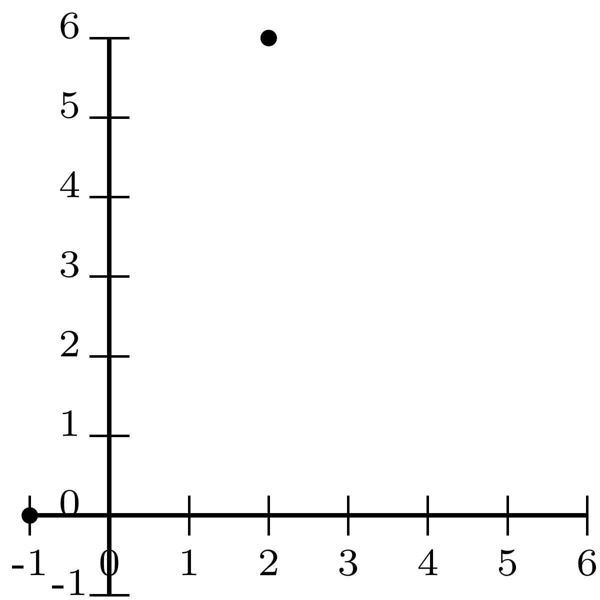
$$y = mx + b$$

3. Substitute x and y values for one of the points and solve for b

4. Check your answer using the x and y values for the other point (Always!)

Example: Find the Equation for a Line

Suppose you have two points (-1,0) and (2,6) and you want to find the equation for the line through those two points.



Example: Find the Equation for a Line

1. The slope is the change in y divided by the change in x:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 0}{2 - (-1)} = \frac{6}{3} = 2$$

2. Set up a formula in the standard form, using the slope you calculated

$$y = 2x + b$$

Example: Find the Equation for a Line

3. Substitute the x and y values for one of the points (2,6) and solve for b

$$6 = 2(2) + b$$

$$6 = 4 + b$$

$$6 - 4 = b \Rightarrow b = 2$$

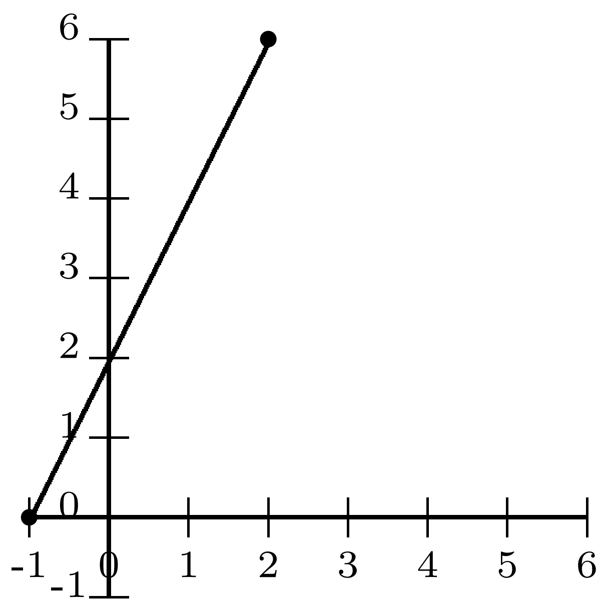
$$y = 2x + 2$$

4. Check your answer using the x and y values for the other point

$$0 = 2(-1) + 2? \text{ Yes!}$$

Example: Find the Equation for a Line

$$y = 2x + 2$$



Another Way to Find the Equation for a Line

We have been using the slope-intercept form of the equation for a line. Another way to find the equation for a line is to use the point-slope method.

$$y - y_1 = m(x - x_1)$$

$m =$ slope

$(x_1, y_1) =$ one point on the line

So, given slope $= 2$ and point $(x_1, y_1) = (-1, 0)$:

$$y - 0 = 2(x - (-1))$$

$$y = 2x + 2$$

Same as before!

Solving Quadratic Equations

You may need to find the solution to a quadratic equation. To do this, you must use the distributive, additive, and multiplicative properties to get the equation into this form:

$$ax^2 + bx + c = 0$$

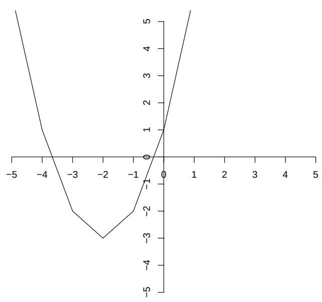
Then you can plug a, b, and c into the following equation, which is called the **quadratic formula**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

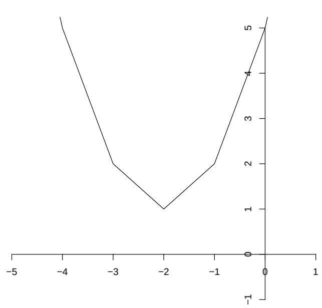
$\sqrt{b^2 - 4ac}$ is called the *discriminant*.

Solving Quadratic Equations

The solution to a quadratic equation specifies where it crosses the x axis. A quadratic equation may have 2 solutions:

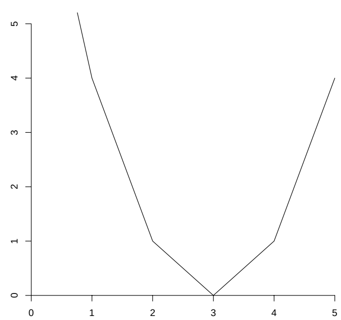


A quadratic equation may have no solutions:



Solving Quadratic Equations

A quadratic equation may have one solution:



Solving Quadratic Equations: Example

$$
\begin{aligned}
3(x^2 + 2x) + 2 &= -x^2 - 6x - 4 \\
3x^2 + 6x + 2 &= -x^2 - 6x - 4 \\
4x^2 + 6x + 2 &= -6x - 4 \\
4x^2 + 6x + 6 &= -6x \\
4x^2 + 12x + 6 &= 0
\end{aligned}
$$

So $a = 4$, $b = 12$ and $c = 6$.

Solving Quadratic Equations: Example

$$a = 4, b = 12, c = 6$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-12 \pm \sqrt{12^2 - 4(4)(6)}}{(2)(4)} \\
x &= \frac{-12 \pm \sqrt{48}}{8} \\
x &= \frac{-12}{8} \pm \frac{\sqrt{48}}{8} \\
x &= -1.5 + 0.8660 = -0.634 \\
x &= -1.5 - 0.8660 = -2.366
\end{aligned}$$

The two solutions are -0.634 and -2.366.

Solving Quadratic Equations: Example

Check the solutions:

$$
\begin{aligned}
4x^2 + 12x + 6 &= 0 \\
4(-0.634)^2 + 12(-0.634) + 6 &= 1.608 - 7.608 + 6 = 0 \\
4(-2.366)^2 + 12(-2.366) + 6 &= 22.392 - 28.392 + 6 = 0
\end{aligned}
$$

Good!

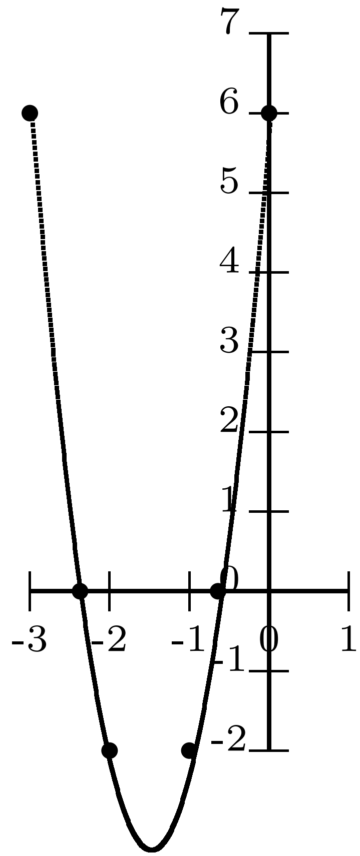
Solving Quadratic Equations: Example

We can graph quadratic equations in a manner similar to that for linear functions:

$$y = 4x^2 + 12x + 6$$

x	y
0	6
-1	-2
-2	-2
-3	6
-0.634	0
-2.366	0

Graphing Quadratic Equations



Solving Quadratic Equations: Example

Recall the 8 animals who received different doses of a drug and whose weight gain was measured. The quadratic equation that best described the relationship between dose and weight gain was:

$$y = 1.13 - 0.41x + 0.17x^2$$

We can use substitution to find the predicted weight gain, given a dose. For example, if we know an animal like these received dose 3, we would predict that the weight gain would be
$1.13 - 0.41(3) + 0.17(3)(3) = 1.43$ dekagrams.

What if we knew the animal had gained 5 dekagrams, and wanted to deduce what the dose had been?

Solving Quadratic Equations: Example

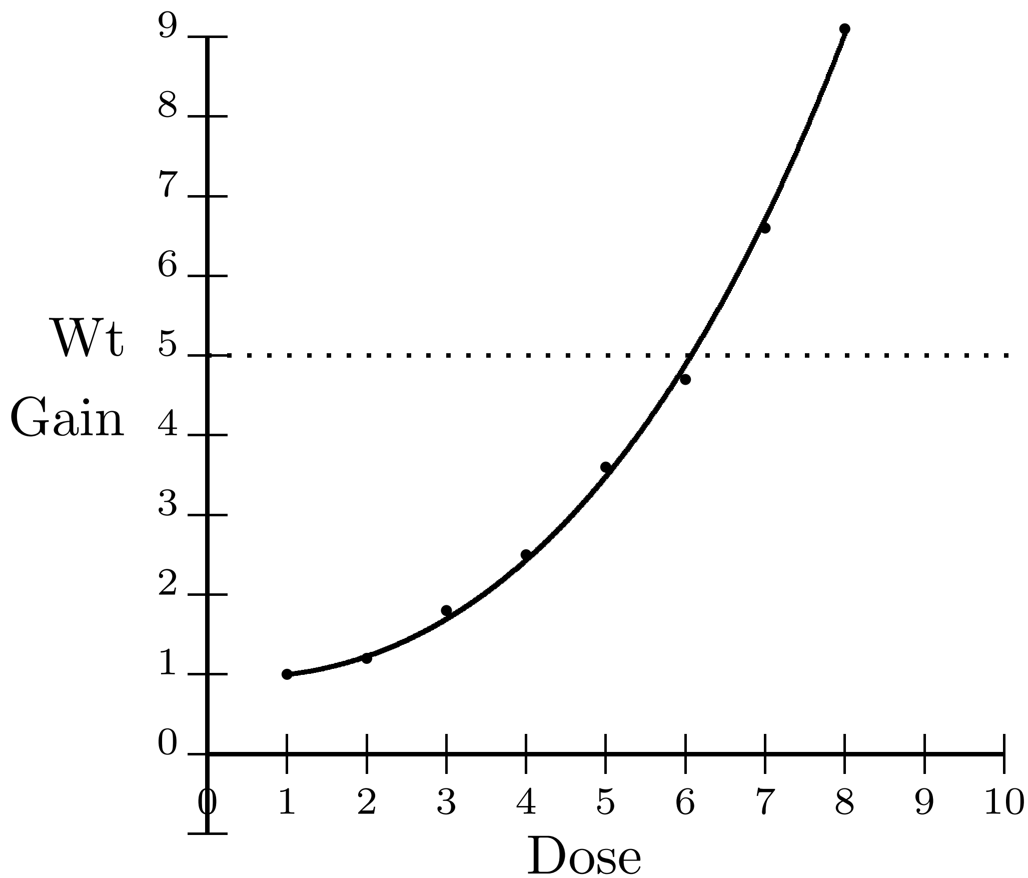
Hint: Before getting tangled up in arithmetic, think about what a reasonable solution might be. Recall:

Dose	Weight Gain (dekagrams)
1	1
2	1.2
3	1.8
4	2.5
5	3.6
6	4.7
7	6.6
8	9.1

If the animal gained 5 dekagrams, the dose was probably between 6 and 7.

Solving Quadratic Equations: Example

Hint: When possible, draw a picture.



Solving Quadratic Equations: Example

If the animal had gained 5 dekagrams, substituting for y,

$$\begin{aligned} y = 5 &= 1.13 - 0.41x + 0.17x^2 \\ 5 - 5 &= 1.13 - 5 - 0.41x + 0.17x^2 \\ 0 &= 0.17x^2 - 0.41x - 3.87 \end{aligned}$$

We can use the quadratic equation with

$$a = 0.17,\ b = -0.41,\ c = -3.87$$

Solving Quadratic Equations: Example

$$
\begin{aligned}
x &= \frac{0.41 \pm \sqrt{(-0.41)^2 - ((4)(0.17)(-3.87))}}{(2)(0.17)} \\
x &= \frac{0.41 \pm \sqrt{0.1681 + 2.6316}}{0.34} \\
x &= \frac{0.41 \pm \sqrt{2.7997}}{0.34} \\
x &= \frac{0.41}{0.34} \pm \frac{1.6732}{0.34} \\
x &= 1.206 \pm 4.921 \\
x &= 6.127 \text{ or } -3.715
\end{aligned}
$$

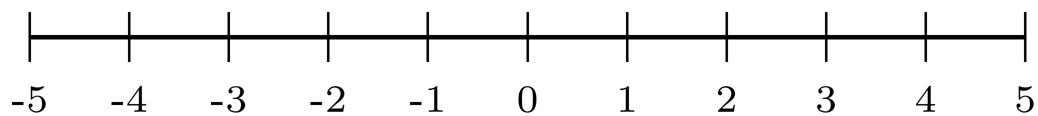
We should go back and check these!

Solving Quadratic Equations: Example

The solutions to the equation are 6.127 and -3.715. Do both of these make sense in the context of our problem? No - animals cannot receive a negative dose! We are only interested in solutions greater than zero. The predicted dose is 6.127. (We can envision a dose bigger than 6, but less than 7.) This agrees with our best estimate before doing the math.

Absolute Value Equations

Recall: **Absolute value** refers to a number's distance from 0 on the real number line.



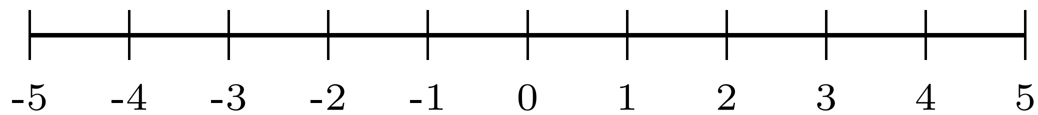
$$|-5| = 5 \qquad |2| = 2 \qquad |0| = 0$$

$|a - b|$ is the absolute value of $(a - b)$, or the distance from a to b

Example:

$$|5 - 3| = 2$$

Absolute Value Equations



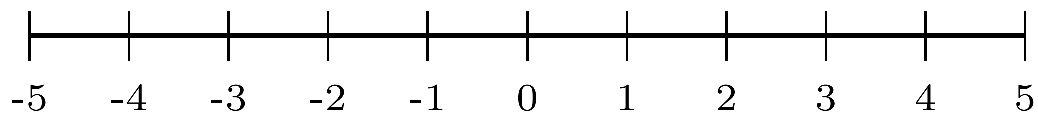
$|a+b|$ can be thought of as $|a-(-b)|$, or the distance between a and $-b$.

$|5+3| = |5-(-3)|$, or the distance betwen 5 and -3, or 8.

$|-a-b| = |-a-(+b)|$, or the distance from $-a$ to b

$|-5-3| = |-5-(+3)| = |-8| = 8$

Solving Absolute Value Equations



$|a| = 5$ means, “What values of a are 5 units away from 0?”

Solutions: 5 and -5

$|a - 3| = 2$ means, “What are the values of a such that the distance between a and 3 is 2?”

We can deduce the answer by finding 3 on the number line and finding the values 2 units away from 3.

Algebraically, we can solve the equations $a - 3 = 2$ and $a - 3 = -2$.

So $a = 5$ and $a = 1$

Solving Absolute Value Equations

Solve	$5 - \|3x + 5\| = 3$	
Subtract 5	$5 - 5 - \|3x + 5\| = 3 - 5$	
Simplify	$-\|3x + 5\| = -2$	
Mult. by -1	$(-1)(-\|3x + 5\|) = (-1)(-2)$	
Simplify	$\|3x + 5\| = 2$	
Solve		
separately	$3x + 5 = 2$	$3x + 5 = -2$
Subtract 5	$3x + 5 - 5 = 2 - 5$	$3x + 5 - 5 = -2 - 5$
Simplify	$3x = -3$	$3x = -7$
Mult. by $\frac{1}{3}$	$(3x)(\frac{1}{3}) = (-3)(\frac{1}{3})$	$(3x)\frac{1}{3} = (-7)\frac{1}{3}$
Solutions	$x = -1$	$x = \frac{-7}{3}$

Solving Absolute Value Equations

Highly recommended: Check your solutions by plugging them back into the original equation:

$$
\begin{aligned}
5 - |3(-1) + 5| &\stackrel{?}{=} 3 \\
5 - |-3 + 5| &\stackrel{?}{=} 3 \\
5 - |2| &\stackrel{?}{=} 5 - 2 = 3 \\
\text{Yes!} & \\
5 - |3(\frac{-7}{3}) + 5| &\stackrel{?}{=} 3 \\
5 - |-7 + 5| &\stackrel{?}{=} 3 \\
5 - |-2| &\stackrel{?}{=} 5 - 2 = 3 \\
\text{Yes!} &
\end{aligned}
$$

Solving Absolute Value Equations

Note that simplifying absolute values should come early in the order of operations, similar to brackets and parentheses.

$$|3x+5| = 2 \text{ is not the same as } |3x| + 5 = 2$$

Example: Using Absolute Values in Statistics

You are studying the relationship between head injury and helmet use among people involved in bicycle accidents (Pagano and Gauvreau, page 311).

You have information for 793 people involved in accidents. Of those, 147 people were wearing helmets and 235 people experienced a head injury. You can make a table that illustrates what you know, and add what you would expect to see if there were no association between helmet use and head injury:

		Wearing Helmet		
		Yes	No	Total
Head	Yes			235
Injury	No			
	Total	147		793

Example: Using Absolute Values in Statistics

Here is the table of **expected** values we just created:

Wearing Helmet		Yes	No	Total
Head	Yes	43.6	191.4	235.0
Injury	No	103.4	454.6	558.0
	Total	147.0	646.0	793.0

When you add the information you have about helmet use among people who experienced a head injury, here is what you **observed**:

Wearing Helmet		Yes	No	Total
Head	Yes	17	218	235
Injury	No	130	428	558
	Total	147	646	793

Example: Using Absolute Values in Statistics

We can use a "test statistic" to see whether the difference between observed and expected values is big enough to signify that wearing helmets is associated with fewer head injuries. Here's what the test statistic looks like:

$$\chi^2 = \sum_{i=1}^{4} \frac{(|O_i - E_i| - 0.5)^2}{E_i}$$

$\sum_{i=1}^{4}$ means "add the first through the 4th" of what comes next

O_i is the observed value for position i, where i is either 1, 2, 3, or 4

E_i is the expected value in the corresponding position

χ^2 (Chi-squared) is the name of this particular test statistic

Example: Using Absolute Values in Statistics

$$
\begin{aligned}
\chi^2 &= \sum_{i=1}^{4} \frac{(|O_i - E_i| - 0.5)^2}{E_i} \\
\chi^2 &= \frac{(|17 - 43.6| - 0.5)^2}{43.6} + \frac{(|130 - 103.4| - 0.5)^2}{103.4} + \\
&\quad \frac{(|218 - 191.4| - 0.5)^2}{191.4} + \frac{(|428 - 454.6| - 0.5)^2}{454.6} \\
&= 15.62 + 6.59 + 3.56 + 1.50 \\
&= 27.27
\end{aligned}
$$

What does this mean? Is there an association between helmet use and head injury? Stay tuned!

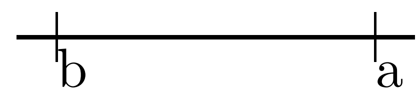
Inequalities

An inequality is like an equation, but it says that two expressions are *not* equal.

$a \neq b$ a is not equal to b

$a < b$ a is less than b

$a > b$ a is greater than b



$a \geq b$ a is greater than or equal to b

$a \leq b$ a is less than or equal to b

Note that “less than” means “to the left on the number line” and “greater than” means “to the right on the number line”.

Example: Using Inequalities

The value of the test statistic, χ^2, that we obtained in the last example was 27.27. We can look this up in a table and see how "statistically significant" this is. The smaller the p-value, the more statistically significant.

0.100	0.050	0.025	0.010	0.001	$\leftarrow p$-value
2.71	3.84	5.02	6.63	10.83	$\leftarrow \chi^2$

As we can see from the table, the p-value gets smaller as the χ^2 values get bigger. Since $27.27 > 10.83$, $p < 0.001$.

We conclude that wearing helmets is associated with lower rates of head injury.

Solving Linear Inequalities

As with equalities, we can use the additive and multiplicative properties of inequality.

One wrinkle: If you **multiply** both sides of an inequality by a negative number, the **direction** of the inequality changes. So multiplying or dividing both sides of an equation by a negative number means you should reverse the inequality.

Example:

$$
\begin{array}{rcll}
4 & < & 7 & \\
(4)(-1) & \overset{?}{<} & (7)(-1) & \text{multiply by -1} \\
-4 & \overset{?}{<} & -7 & \text{Not true!} \\
-4 & > & -7 & \text{reverse direction of inequality}
\end{array}
$$

Solving Linear Inequalities: Example

Solve:	$x + 3 > 4x + 6$
add -4x to both sides	$x - 4x + 3 > 4x - 4x + 6$
combine like terms	$-3x + 3 > 6$
add -3 to both sides	$-3x + 3 - 3 > 6 - 3$
	$-3x > 3$
multiply by $\frac{-1}{3}$	$(-3x)(\frac{-1}{3}) < (3)(\frac{-1}{3})$
& change direction of	
inequality	$x < -1$

Check: Does $x = -2$ work? Does $x = -3$ work?