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Optimal orbits of hyperbolic systems

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Abstract. Given a dynamical system and a function f from the state space to the real numbers, an optimal orbit for f is an orbit over which the time average of f is maximal. In this paper we consider some basic mathematical properties of optimal orbits: existence, sensitivity to perturbations of f, and approximability by periodic orbits with low period. For hyperbolic systems, we conjecture that for (topologically) generic smooth functions, there exists an optimal periodic orbit. In support of this conjecture, we prove that optimal periodic orbits are insensitive to small C^1 perturbations of f, while the optimality of a non-periodic orbit can be destroyed by arbitrarily small C^1 perturbations. In case there is no optimal periodic orbit for a given f, we discuss the question of how fast the maximum average over orbits of period at most p must converge to the optimal average, as p increases.

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1. Introduction

Recently, the following optimization problem [1, 2] has been posed: which orbit(s) on a given attractor yield the largest time average of a given smooth function f? One motivation for this question is as follows. A popular method of 'controlling chaos' [3, 4] involves using small perturbations to stabilize the system near an unstable periodic orbit that is embedded in the chaotic attractor. A typical chaotic attractor contains infinitely many periodic orbits. Which one should be used in a given application? Here is a natural way to select among them. Let f be a smooth (C^1) function from the phase space to $\boldsymbol{R}$ that measures the 'performance' of the output of the system at a given time. Then, choose an orbit (which may not be unique) that maximizes the time average of f, i.e. an orbit that has the best average performance. If such an orbit exists, we call it an optimal orbit. More generally we can consider optimization over all orbits within the attractor. Is the optimum average realized by an unstable periodic orbit? If so, will this orbit have low or high period?

An important feature of chaotic systems is ergodicity. It is often assumed, though rarely proved, that a chaotic attractor has a *natural measure*, i.e., for typical initial conditions (in the sense that the exceptional subset of the basin of attraction has Lebesgue measure zero), the invariant measures generated by their trajectories are the same. Nonetheless, there will be many other orbits that generate different invariant measures. The invariant measure generated by an optimal orbit will not be the natural measure, except in very special cases (see section 3).

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Hunt and Ott [1, 2] investigate numerically some one- and two-dimensional maps and several one-parameter families of performance functions. Considering unstable periodic orbits up to period 24, they observe that in most cases (with respect to Lebesgue measure on the parameter space), the average is optimized by an orbit with low period. Furthermore, they argue that although there are cases in which it seems that for a set of parameters with Lebesgue measure zero, no optimal periodic orbits can be found, the corresponding optimal non-periodic orbits are special in the sense that their limit sets have zero topological entropy and zero fractal dimension (in particular, they are not dense in the attractor). They conjecture [1, 2] that, typically, there exists an optimal periodic orbit for almost every parameter value with respect to Lebesque measure. We formulate a similar but more precise conjecture below, replacing its measure-theoretic aspect with the notion of topological genericity, which calls a set *generic* if it contains a countable intersection of open dense sets.

The purpose of this paper is to mathematically establish some fundamental properties of optimal orbits: existence, sensitivity to parameter perturbations, and approximability by periodic orbits with low period. We consider only discrete-time systems (maps), and we first prove existence assuming only continuity of the map and the performance function. Thereafter, we assume that these functions are smooth and that the dynamics are hyperbolic, in the sense that the map either is a diffeomorphism satisfying axiom A or is noninvertible and uniformly expanding. In sections 3 and 4 we prove some results related to the following conjecture.

Conjecture 1.1. *For an axiom A or uniformly expanding system T and a (topologically) generic smooth function f, there exists an optimal periodic orbit.*

In applications it is also of considerable interest to know how close one can come to the optimal average by considering only periodic orbits with low period. We formulate this question in terms of how fast, in the worst case, the maximum average over orbits of period at most p must converge to the optimal average, as $p \to \infty$.

Question 1.2. *For an axiom A or uniformly expanding system T and a smooth function f, let d_p be the difference between the optimal average of f and its maximum average over periodic orbits with period at most p. How fast can one prove that $d_p \to 0$ as $p \to \infty$?*

In corollary 3.4 we show that d_p must decay faster than a power of p, but in all of the examples we have been able to analyze, the decay is much faster.

During the course of preparing this paper, we learned that similar problems have been studied in different contexts. In particular, in his study of Lagrangian flows, Mañé [5] proves that for a generic Lagrangian L, there is a unique invariant probability measure that minimizes the average of L. In that paper, he also poses the following question: is it true that for a generic Lagrangian L, this minimizing measure is supported on a periodic orbit? (See also [6, 7] for further results on minimizing measures for Lagrangian systems.) Bousch [8] considers the optimal averages of $\cos(2\pi(x - \theta))$, where θ is a parameter, over orbits of the doubling map $x \mapsto 2x$ (mod 1). He shows that for all θ, there is a unique optimal invariant measure, whose support is contained in a semicircle. (It follows that the support must have Hausdorff dimension zero [9].) Further, he shows that for almost every θ (with respect to Lebesgue measure), the optimizing invariant measure is supported on a periodic orbit. (These results were conjectured by Jenkinson [10, 11], and the latter result was conjectured independently by Hunt and Ott [1, 2].) More generally, Contreras *et al* [12] consider C^1 expanding maps of the circle and the class of C^α performance functions f, where $0 < \alpha < 1$. They prove that for (topologically) generic f, there is a unique maximizing measure. Furthermore, within the subset of C^α functions consisting of the closure of the union of C^β functions for all $\beta > \alpha$, they show that generically (in the C^α topology) the maximizing measure is supported on a periodic orbit. However, this subset is not dense in the set of all C^α functions, so the analogue of conjecture 1.1 for C^α functions is still open.

In section 2 we consider the existence of optimal orbits and invariant measures. We allow all orbits, even those for which the performance average does not exist, using a 'limsup' average in that case. We show for continuous dynamical systems and continuous performance functions, there must exist an optimal orbit that is measure-recurrent (i.e., contained in the support of the invariant measure that it generates).

In section 3 we restrict our attention to axiom A and uniformly expanding systems and Lipschitz performance functions. We show, using the quantitative versions of the standard shadowing and closing lemmas for hyperbolic systems, that either an optimal periodic orbit exists, or every optimal orbit has no periodic points in its closure. A modification of this argument shows that there always exists an optimal orbit supported on a minimal set. We also show that the optimal average can always be approximated algebraically well by averages over optimal orbits with increasing periods; this is a partial answer to question 1.2.

In section 4 we address conjecture 1.1 more directly by showing (still focusing on axiom A and uniformly expanding systems and Lipschitz performance functions) that optimal periodic orbits are more robust than non-periodic ones, in the following sense. We prove that each periodic orbit is optimal for some open set of Lipschitz performance functions, but if a non-periodic, measure-recurrent orbit is optimal for some Lipschitz performance function f, then there exist arbitrarily small Lipschitz perturbations of f for which that orbit is not optimal. We also indicate how to extend these results to the C^1 topology.

Finally, in section 5 we summarize and further discuss the main results in this paper.

2. Existence

In this section we establish the existence of optimal orbits in a general setting. Though the discussion here could be simplified by discussing only optimal invariant measures, our techniques in the following sections require the analysis of specific orbits.

We begin with a precise definition of an optimal orbit.

Definition 2.1. *Let M be a compact smooth Riemannian manifold, $T : M \hookleftarrow$ be a continuous map, and $f(x)$ be a real-valued continuous function on M. Let*

$$S_N(x) = \frac{1}{N}\sum_{k=1}^{N} f(T^k x),$$

and let $\langle f\rangle(x) = \lim_{N\to\infty} S_N(x)$ if the limit exists. If $\langle f\rangle(x_0)$ is defined, and for each $x \in M$, $\langle f\rangle(x_0) \geqslant \limsup_{N\to\infty} S_N(x)$, then the orbit of x_0 is called an optimal orbit.

From the mathematical point of view, a fundamental question is: Does an optimal orbit always exist?

For any $x \in M$, if the weak limit of $\frac{1}{N}\sum_{k=1}^{N}\delta_{T^k x}$ exists as $N \to \infty$, where δ_x is the Dirac measure concentrated at x exists, then we say x *generates* an invariant measure and this limit measure is the measure that is *generated* by x. We say a point $x \in M$ and its orbit are *measure recurrent* if (i) x generates an invariant measure; and (ii) x lies in the support of the measure generated by x. The ω-limit set of a point $x \in M$ is defined as $\omega(x) = \bigcap_{n=0}^{\infty}\overline{\bigcup_{k=n}^{\infty}\{T^k x\}}$. We observe that if x is measure recurrent, then $\omega(x)$ is equal to the support of the measure generated by x. Now we give an affirmative answer to our question about the existence of optimal orbits in the following proposition.

Proposition 2.2. *Under the hypothesis of definition 2.1, there always exists a measure-recurrent optimal orbit.*

In the remainder of this section we prove proposition 2.2.

Lemma 2.3. *For every $x \in M$, there exists an invariant measure μ_x such that $\int f\,\mathrm{d}\mu_x = \limsup_{N\to\infty} S_N(x)$.*

Proof. There exists a subsequence $\{S_{N_i}(x)\}$ that converges to $\limsup_{N\to\infty} S_N(x)$. Define $\mu_{x,N_i} = \frac{1}{N_i}\sum_{k=1}^{N_i}\delta_{T^k x}$. Since the space of Borel probability measures is compact in weak topology, there exists a subsequence of $\{\mu_{x,N_i}\}$ that converges weakly to a probability measure μ_x. It is easy to check that μ_x is invariant. Moreover, $\int f\,\mathrm{d}\mu_x = \limsup_{N\to\infty} S_N(x)$. □

Lemma 2.4. *There exists an invariant measure ν such that*

$$\int f\,\mathrm{d}\nu \geqslant \limsup_{N\to\infty} S_N(x), \qquad \textit{for every} \quad x \in M.$$

Proof. Let $\beta = \sup_x \limsup_{N\to\infty} S_N(x)$, and let $\{x_i\}$ be a sequence for which $\limsup_{N\to\infty} S_N(x_i) \to \beta$ as $i \to \infty$. Let μ_{x_i} be as in lemma 2.3. There exists a subsequence of $\{\mu_{x_i}\}$ that converges weakly to an invariant measure ν, whence $\int f\,\mathrm{d}\nu = \beta$. □

Proof of proposition 2.2. Of all the invariant measures that satisfy lemma 2.4, there is at least one ergodic measure μ (by the ergodic decomposition theorem; see e.g. [13, section 4.1]). And by the Birkhoff ergodic theorem, almost every point with respect to μ generates μ. Therefore, we can select x from the support of μ. Then x is a measure-recurrent optimal orbit. □

3. Minimality

Given a chaotic dynamical system, we cannot expect every performance function from a given class, such as C^1, to have an optimal periodic orbit. For example, given any compact invariant set S that contains no periodic orbits, one can construct f to be zero on S and negative at all other points. We consider, in this section, the properties that an optimal non-periodic orbit must have in the case that no optimal periodic orbit exists. For example, is it possible that such an orbit is dense in M? We show in proposition 3.3 that the answer to this question is essentially no for axiom A and uniformly expanding systems. In fact, proposition 3.3 implies that a dense orbit can be optimal only if the average $\langle f\rangle$ is the same for all orbits on M.

Let T be a diffeomorphism, and Λ be a compact invariant set. We say Λ is *hyperbolic* if there exist a continuous splitting $T_\Lambda M = E_1 \oplus E_2$ and positive constants C, λ and κ with $\lambda < 1 < \kappa$ such that (i) $DT(E_i) = E_i, i = 1, 2$; (ii) for all $v \in E_1$ and $n \geqslant 0$, $|DT^{-n}(v)| \leqslant C\kappa^{-n}|v|$; and (iii) for all $v \in E_2$ and $n \geqslant 0$, $|DT^n(v)| \leqslant C\lambda^n|v|$. We say a point $x \in M$ is *non-wandering* if for every neighbourhood U of x, there exists $n > 0$ such that $T^n(U)\bigcap U \neq \emptyset$. Let Ω be the set of non-wandering points for T. We say T satisfies *axiom A* if Ω is hyperbolic and periodic orbits are dense in Ω. Notice that if x is measure recurrent, then $\overline{\{T^i x\}_{i=0}^{\infty}} = \omega(x) \subset \Omega$, and therefore T is hyperbolic on $\overline{\{T^i x\}_{i=0}^{\infty}}$.

Standard definitions of hyperbolicity do not allow T to be non-invertible. However, it is well known that certain non-invertible maps (like the $2x$ mod 1 map) share many properties with hyperbolic diffeomorphisms. In particular, we find that for our purpose, we can include with axiom A diffeomorphisms, those non-invertible C^1 maps that are *uniformly expanding*, i.e., there exists $\kappa > 1$ such that $|DT(x)v| \geqslant \kappa|v|$ for all points x and tangent vectors v. (Notice that we do not allow piecewise smooth expanding maps like tent maps in this definition.)

Standing hypothesis. *In the rest of this paper we assume T is either an axiom A diffeomorphism or non-invertible uniformly expanding map. Also, we require the performance function f to be Lipschitz continuous.*

For $\delta > 0$, we say an orbit $\{T^i x\}_{i=0}^{\infty}$ *comes within* δ of a periodic orbit $\{y = T^p y, Ty, \ldots, T^{p-1} y\}$ if there exists $m \geqslant 0$ such that $d(T^{m+i}x, T^i y) \leqslant \delta$ (i.e. the distance between these two points in Riemannian metric) for $0 \leqslant i < p$.

The main technical result in this section is the following proposition.

Proposition 3.1. *Let $\{T^i x\}_{i=0}^{\infty}$ be a measure-recurrent optimal orbit for f. Then there exist c_0 and $c_1 > 0$ such that if $\delta > 0$ is sufficiently small, the following statements hold.*

(a) If for some m and p, $d(T^m x, T^{m+p}x) < \delta/c_0$, then there exists a period-p orbit $\{T^i y\}_{i=0}^{p-1}$ such that $\max_{0 \leqslant i < p} d(T^{m+i}x, T^i y) \leqslant \delta$; in other words, $\{T^{m+i}x\}_{i=0}^{p-1}$ is δ-shadowed (see definition 3.6) by $\{T^i y\}_{i=0}^{p-1}$.

(b) If $\{T^i y\}_{i=0}^{p-1}$ is a period-p orbit and $\{T^i x\}$ comes within δ of $\{T^i y\}$, then $\langle f\rangle(x) - c_1\delta/p < \langle f\rangle(y) \leqslant \langle f\rangle(x)$.

We prove proposition 3.1 later. Now we present a few consequences of this proposition.

Corollary 3.2. *Let $\{T^i x\}_{i=0}^{\infty}$ be a measure-recurrent optimal orbit for f. Then for every periodic point $y \in \omega(x)$, $\langle f\rangle(y) = \langle f\rangle(x)$.*

Proof. Let $\{T^i y\} \subset \omega(x)$ be a period-p orbit. Then there exists $n_k \to \infty$ such that $\lim_{k\to\infty} T^{n_k} x = y$. It follows that $\{T^i x\}$ comes within δ of y for all $\delta > 0$, so by proposition 3.1(b), $\langle f\rangle(y) = \langle f\rangle(x)$. □

A consequence of corollary 3.2 is that if there are no optimal periodic orbits, then every measure-recurrent optimal orbit has the property that its ω-limit set contains no periodic orbits. Moreover, using similar reasoning, we can prove the following proposition.

Proposition 3.3. *For each f, there exists a measure-recurrent optimal orbit whose closure is a minimal invariant set. (By 'minimal' we mean that the set has no proper, closed, nonempty subset that is T-invariant.)*

If there are no optimal periodic orbits, or there exists only an optimal periodic orbit with a high period, are there any periodic orbits with low period that yield an 'approximately optimal' average? When controlling a chaotic system with small perturbations, as discussed in the introduction, it may only be practical to consider low-period orbits. Though, of course, it is impossible to be precise in an abstract setting about how close to the optimum performance function average one can come using only 'low'-period orbits, we can consider the rate of convergence to the optimal average as the maximum period increases. This is the point of question 1.2 in the introduction. In the following corollary of proposition 3.1 we answer this question, partially, in terms of the 'worst case' rate of recurrence of the optimal orbit. In particular, for an arbitrary trajectory $\{T^i x\}$ and a positive integer p, we define $\epsilon_p(x)$ as the closest recurrence in the trajectory within p iterations, i.e., $\epsilon_p(x) = \inf\{|T^i x - T^j x| : 0 < |i - j| \leqslant p\}$. (Notice that we consider recurrences between any two points in the trajectory, not just recurrences to the initial condition.)

Corollary 3.4. *Let $\{T^i x\}_{i=0}^{\infty}$ be a measure-recurrent optimal orbit for f. Then there exists $C > 0$ such that for all p, there exists a periodic orbit $\{T^i y_p\}$ of period at most p such that $\langle f\rangle(y_p) > \langle f\rangle(x) - C\epsilon_p(x)$. Furthermore, we have $\langle f\rangle(y_p) > \langle f\rangle(x) - Cp^{-1/m}$, where m is the dimension of the ambient manifold M.*

Proof. Let $\epsilon_p = \epsilon_p(x)$. There exist integers r and s such that $0 < s - r \leqslant p$ and $d(T^r x, T^s x) < 2\epsilon_p$. From proposition 3.1(a), $\{T^i x\}_{i=r}^{s-1}$ can be $(2c_0\epsilon_p)$-shadowed by a period-$(s-r)$ orbit $\{T^i y_p\}$. From proposition 3.1(b),

$$\langle f\rangle(y_p) > \langle f\rangle(x) - 2c_1c_0\epsilon_p(s-r)^{-1} \geqslant \langle f\rangle(x) - 2c_1c_0\epsilon_p.$$

There exists $c' > 0$ such that for every $p > 0$ and every set of p points in M, the smallest separation between any two points is bounded by $c'p^{-1/m}$. Thus, $\epsilon_p \leqslant c'p^{-1/m}$. The previous equation yields

$$\langle f\rangle(y_p) > \langle f\rangle(x) - 2c'c_1c_0p^{-\frac{1}{m}}.$$

Let $C = \max\{2c_1c_0c', 2c_1c_0\}$ to complete the proof. □

Remark 3.5. *In terms of question 1.2, we have shown that $d_p \leqslant Cp^{-1/m}$. One may be able to improve this estimate by not throwing away the term $(s-r)^{-1}$ above, but we think that the best opportunity for improvement lies in using the dynamics of T to improve the geometric estimate $\epsilon_p \leqslant c'p^{-1/m}$. Indeed, for the non-periodic optimal orbits in the example studied in [1,2,8,10,11], one can show that ϵ_p decreases exponentially, i.e., $\epsilon_p \leqslant c\exp(-\alpha p)$. What is the 'worst case' for the rate of decrease of ϵ_p? We think that this is an interesting but difficult problem.*

In the remainder of this section we prove proposition 3.1, but first we need some preparations.

Definition 3.6. *A sequence $\bar{x} = \{x_i\}_{i=a}^{b} \subset M$ is called a* δ-pseudo-orbit *if $d(Tx_i, x_{i+1}) < \delta$ for all $a \leqslant i < b$. We say $\bar{x}$ is* ϵ-shadowed *by the orbit of $x \in M$ if $d(T^i x, x_i) < \epsilon$ for all $a \leqslant i < b$.*

Lemma 3.7 (Anosov–Bowen shadowing lemma). *If Λ is a compact hyperbolic invariant set, then there exist $c_0 > 0$ and an open neighbourhood $U \supset \Lambda$ such that for every $\delta > 0$, every δ-pseudo-orbit in U is $(c_0\delta)$-shadowable.*

Our statement of the shadowing lemma is stronger than the standard version (for example, see [13, section 18.1]) in that it says that the shadowing distance is bounded by the noise level δ multiplied by a constant c_0. However, this can be easily deduced from the proof of the shadowing lemma.

The following lemma strengthens the usual version of the Anosov closing lemma.

Lemma 3.8 (see [13], section 6.4). *Let Λ be a compact hyperbolic invariant set for T. Then there exist an open neighbourhood $U \supset \Lambda$, positive numbers c_0, ϵ_0, and $\lambda \in (0,1)$ such that:*

(a) if $T^i x \in U$, for $i = 0, \ldots, n$ and $d(T^n x, x) < \epsilon_0$ then there exists a periodic point y with $T^n y = y$ such that

$$d(T^i y, T^i x) < c_0\lambda^{\min(i,n-i)}\, d(T^n x, x);$$

(b) for every positive integer n, $x \in \Lambda$ and $y \in U$ such that $d(T^i x, T^i y) < \epsilon_0$, for $i = 0, \ldots, n$,

$$d(T^i x, T^i y) < c_0\lambda^{\min(i,n-i)}(d(x,y) + d(T^n x, T^n y)).$$

Remark 3.9. *If T is non-invertible and uniformly expanding, then both the shadowing and closing lemmas (in the form of lemmas 3.7 and 3.8) still hold, with the replacement of the hyperbolic invariant set Λ by the whole manifold M.*

In the remainder of this section, we fix our choices of the constants c_0, ϵ_0 and λ so that lemmas 3.7 and 3.8 hold.

Lemma 3.10. *Let $\{T^i x\}_{i=0}^{\infty}$ be a measure-recurrent optimal orbit for f. Then there exists $c_1' > 0$ such that if $\delta > 0$ is sufficiently small and $d(T^{m+p}x, T^m x) < \delta$, then*

$$\left|\frac{1}{p}\sum_{i=0}^{p-1} f(T^{m+i}x) - \langle f\rangle(x)\right| < \frac{c_1'\delta}{p}.$$

Proof. Let $L > 0$ be a Lipschitz constant for f. From lemma 3.8(a), there exists a periodic point $y' \in M$ with $T^p y' = y'$ such that $d(T^{m+i}x, T^i y') < c_0\lambda^{\min(i,p-i)}\delta$, for $0 \leqslant i \leqslant p$. Therefore,

$$\left|\frac{1}{p}\sum_{i=0}^{p-1} f(T^{m+i}x) - \langle f\rangle(y')\right| \leqslant \frac{1}{p}\sum_{i=0}^{p-1} |f(T^{m+i}x) - f(T^i y')| \leqslant \frac{L}{p}\sum_{i=0}^{p-1} d(T^{m+i}x, T^i y')$$
$$< \frac{L}{p}\sum_{i=0}^{p-1} c_0\lambda^{\min(i,p-i)}\delta \leqslant \frac{2c_0 L\delta}{(1-\lambda)p}. \tag{3.1}$$

Let $\hat{c}_1 = 2c_0L/(1-\lambda)$. Then (3.1) yields

$$\frac{1}{p}\sum_{i=m}^{m+p-1} f(T^i x) < \langle f\rangle(y') + \frac{\hat{c}_1\delta}{p} \leqslant \langle f\rangle(x) + \frac{\hat{c}_1\delta}{p}.$$

It remains to be shown that there exists $c_1' > \hat{c}_1$ such that

$$\frac{1}{p}\sum_{i=m}^{m+p-1} f(T^i x) > \langle f\rangle(x) - \frac{c_1'\delta}{p}.$$

We begin by constructing a pseudo-orbit as follows. Assume $\|DT\| \leqslant K$ and $K > 1$. Let k_1 be the smallest integer such that $d(T^{k_1}x, T^m x) < \delta/K^p$. Given k_j, let k_{j+1} be the smallest integer such that $d(T^{k_{j+1}}x, T^m x) < \delta/K^p$ and $k_{j+1} - k_j \geqslant p$. Throw out all the pieces $\{T^{k_j}x, T^{k_j+1}x, \ldots, T^{k_j+p-1}x\}$ from $\{T^i x\}_{i=0}^{\infty}$. Renumber the remaining sequence, and write it as $\{x_i'\}_{i=0}^{\infty}$. Since $d(T^{m+p}x, T^m x) < \delta$,

$$d(T^{k_j+p}x, T^{k_j}x) \leqslant d(T^{k_j+p}x, T^{m+p}x) + d(T^{k_j}x, T^m x) + d(T^{m+p}x, T^m x)$$
$$< K^p\frac{\delta}{K^p} + \frac{\delta}{K^p} + \delta < 3\delta.$$

Thus, $\{x_i'\}_{i=0}^{\infty}$ is a (3δ)-pseudo-orbit. By lemma 3.7, there exists $c_0 > 0$ (which depends only on T) such that this pseudo-orbit is $(3c_0\delta)$-shadowed by a true orbit $\{T^i z\}_{i=0}^{\infty}$.

The sequence $\{x_i'\}_{i=0}^{\infty}$ is made up of segments $\{T^i x\}_{i=k_j+p}^{k_{j+1}-1}$, where each segment is a true orbit. Let $T^{k_j+p}x = x_{i_j}'$. Then $\{T^i x\}_{i=k_j+p}^{k_{j+1}-1}$ is $(3c_0\delta)$-shadowed by $\{T^i z\}_{i=i_j}^{i_{j+1}-1}$. Let $\ell_j = i_{j+1} - i_j - 1$. Then from lemma 3.8(b),

$$\sum_{i=0}^{\ell_j} d(T^{i_j+i}z, T^{k_j+p+i}x) \leqslant \sum_{i=0}^{\ell_j} c_0\lambda^{\min(i,\ell_j-i)}(d(T^{i_j}z, T^{k_j+p}x) + d(T^{i_{j+1}-1}z, T^{k_{j+1}-1}x))$$
$$\leqslant \sum_{i=0}^{\ell_j} c_0\lambda^{\min(i,\ell_j-i)}6c_0\delta \leqslant \frac{12c_0^2\delta}{1-\lambda}. \tag{3.2}$$

Let

$$\Theta = \sup_{1\leqslant j<\infty} \frac{1}{p}\sum_{i=k_j}^{k_j+p-1} f(T^i x). \tag{3.3}$$

Then:

$$\begin{aligned}\Theta &\leqslant \sup_{1\leqslant j<\infty} \frac{1}{p}\left[\sum_{i=0}^{p-1} f(T^{m+i}x) + L\sum_{i=0}^{p-1} d(T^{k_j+i}x, T^{m+i}x)\right]\\ &< \frac{1}{p}\sum_{i=0}^{p-1} f(T^{m+i}x) + \frac{L}{p}\sum_{i=0}^{p-1} K^{i-p}\delta,\\ &< \frac{1}{p}\sum_{i=0}^{p-1} f(T^{m+i}x) + \frac{L\delta}{(K-1)p},\end{aligned} \tag{3.4}$$

where the second step follows by the definition of k_j, i.e. $d(T^{k_j}x, T^m x) < \delta/K^p$.

For each i, there exists a non-negative integer q_i, such that $x_i' = T^{i+q_i p}x$. To bound Θ from below, observe that

$$\begin{aligned}\langle f\rangle(x) &= \lim_{N\to\infty} \frac{1}{q_N p + N + 1}\left[\sum_{i=0}^{N} f(x_i') + \sum_{j=1}^{q_N}\sum_{i=k_j}^{k_j+p-1} f(T^i x)\right]\\ &\leqslant \frac{1}{q_N p + N + 1}\liminf_{N\to\infty}\left[\sum_{i=0}^{N} f(x_i') + q_N p\Theta\right],\end{aligned}$$

or in other words,

$$\liminf_{N\to\infty} \frac{1}{q_N p + N + 1}\left[\sum_{i=0}^{N} f(x_i') - N\langle f\rangle(x) + q_N p(\Theta - \langle f\rangle(x))\right] \geqslant 0.$$

Define the frequency for those segments that are thrown out as $\Phi = \liminf_{N\to\infty} q_N p/(q_N p + N + 1)$. Then $\Phi \geqslant \mu_x(B(T^m x, \delta/K^p)) > 0$, where $B(T^m x, \delta/K^p)$ denotes the δ/K^p-neighbourhood of $T^m x$. It follows that

$$\liminf_{N\to\infty} \frac{1}{q_N p}\left[\sum_{i=0}^{N} f(x_i') - N\langle f\rangle(x) + q_N p(\Theta - \langle f\rangle(x))\right] \geqslant 0.$$

Thus

$$\begin{aligned}\Theta - \langle f\rangle(x) &\geqslant -\liminf_{N\to\infty} \frac{1}{q_N p}\left[\sum_{i=0}^{N} f(x_i') - N\langle f\rangle(x)\right]\\ &= \limsup_{N\to\infty} \frac{1}{q_N p}\left[N\langle f\rangle(x) - NS_N(z) + \sum_{i=0}^{N}(f(T^i z) - f(x_i'))\right]\\ &\geqslant \limsup_{N\to\infty} \frac{N}{q_N p}[\langle f\rangle(x) - S_N(z)] - \frac{12Lc_0^2\delta}{(1-\lambda)p}, \text{ from (3.2)}\\ &\geqslant \frac{-12Lc_0^2\delta}{(1-\lambda)p}.\end{aligned} \tag{3.5}$$

Combining equations (3.4) and (3.5) yields

$$\frac{1}{p}\sum_{i=0}^{p-1} f(T^{m+i}x) > \Theta - \frac{L\delta}{(K-1)p} \geqslant \langle f\rangle(x) - \frac{12Lc_0^2\delta}{(1-\lambda)p} - \frac{L\delta}{(K-1)p}.$$

Redefine c_1' properly to complete the proof. □

Proof of proposition 3.1. Statement (a) follows immediately from lemma 3.8(a). Thus we only need to prove statement (b). Let y be a periodic point such that $T^p y = y$ and $\{T^i x\}$ comes within δ of the $\{T^i y\}$. Then there exists $m > 0$ such that $d(T^{m+i}x, T^i y) \leqslant \delta$, for $0 \leqslant i < p$. Notice that

$$d(T^{m+p}x, T^m x) < d(T^{m+p}x, y) + d(T^m x, y) \leqslant (K+1)\delta.$$

From lemma 3.8(b), we get

$$d(T^{m+i}x, T^i y) \leqslant 2c_0(K+1)\lambda^{\min(i, p-1-i)}\delta.$$

Thus

$$\left|\frac{1}{p}\sum_{i=m}^{m+p-1} f(T^i x) - \langle f\rangle(y)\right| \leqslant \frac{L}{p}\sum_{i=0}^{p-1} d(T^{m+i}x, T^i y) \leqslant \frac{4c_0 L(K+1)\delta}{p(1-\lambda)}.$$

From lemma 3.10, we have $\langle f\rangle(x) - \sum_{i=m}^{m+p-1} f(T^i x)/p < c_1'(K+1)\delta/p$. Therefore,

$$\langle f\rangle(x) - \langle f\rangle(y) < \frac{c_1'(K+1)\delta}{p} + \frac{4c_0 L(K+1)\delta}{p(1-\lambda)}.$$

Define $c_1 = c_1'(K+1) + 4c_0 L(K+1)/(1-\lambda)$ to complete the proof of statement (b). □

4. Stability of optimal orbits

In this section, we continue our discussion of optimal orbits in the case where our standing hypothesis in the previous section is satisfied. Our emphasis is on the stability of optimal orbits under small perturbations of f, i.e., whether these orbits remain optimal when f is slightly perturbed. To simplify the exposition, we consider perturbations in the space of Lipschitz functions, but we point out how the results here can be extended to the space of C^1 functions as well.

In [14], Hunt and Yorke consider the local extrema of non-differentiable curves given by the basin boundaries of a class of C^1 cylinder maps. They find that for an open set of maps, the local extrema occur at eventually periodic points, while if there is a non-periodic extremum, then there exist arbitrarily small C^1 perturbations of the map that destroy its extremality. In this section, we prove similar results for optimal orbits.

The topology of the space of Lipschitz functions is given by the norm

$$\|f\|_{Lip} = \max_{x\in M}|f(x)| + \sup_{\substack{x,y\in M\\ x\neq y}}\frac{|f(x) - f(y)|}{|d(x, y)|}.$$

For optimal periodic orbits, we make the following observation.

Proposition 4.1. *For every periodic orbit $\{T^i y\}$ there exists an open set G of Lipschitz continuous functions such that for every $f \in G$, $\{T^i y\}$ is optimal. Moreover, it is the unique measure-recurrent optimal orbit.*

Proof. We begin by constructing G. Let p be the period of y. Let γ be the smallest separation of any two points in the y-orbit, i.e., $\gamma = \min_{0\leqslant i<j<p}\{|T^i y - T^j y|\}$. Define $f_0(x) = 1 - d(x, \mathcal{O}(y))$, where $\mathcal{O}(y)$ represents the y-orbit and $d(x, \mathcal{O}(y)) = \min_{0\leqslant i<p} d(x, T^i y)$. Then f_0 is a Lipschitz function and 1 is its Lipschitz constant. Clearly $\{T^i y\}$ is optimal for f_0 and $\langle f_0\rangle(y) = 1$. Let $\epsilon > 0$ be a small number to be determined later. Define $G = \{f_0 + g : \|g\|_{Lip} < \epsilon\}$.

Now we prove, by contradiction, that if ϵ is sufficiently small, then $\{T^i y\}$ is the unique measure-recurrent optimal orbit. From proposition 2.2, for each $f \in G$, there exists

$\{T^i x\} \subset [0, 1]$ that is optimal and measure recurrent. Suppose the orbit $\{T^i x\}$ is different (as a set) from $\{T^i y\}$.

Let $K > 1$ be large enough that $\|DT\| \leqslant K$. Notice that axiom A and uniformly expanding systems are *expansive* [13, section 6.4], i.e., there exists $\sigma > 0$ (which is called an *expansivity constant*) such that for any two points $w, z \in M$, if $d(T^n w, T^n z) < \sigma$ for all $n \in \boldsymbol{N}$, then $\lim_{n\to\infty} d(T^n w, T^n z) = 0$. Let σ be an expansivity constant. Define $\tau = \min(\sigma, \gamma/(3K))$. The set of points z with $d(z, \mathcal{O}(y)) > \tau$ has positive measure with respect to the measure μ_x generated by x. This observation will be used later.

The trajectory $\{T^i x\}$ can be divided into infinitely many segments such that each segment $\{T^m x, \ldots, T^{m+n} x\}$ belongs to one of the following classes: (i) $n = p-1$ and there exists $k > 0$ such that $d(T^{m+i} x, T^{k+i} y) < \tau$ for $0 \leqslant i \leqslant n$; and (ii) $n = 0$ and $d(T^m x, \mathcal{O}(y)) \geqslant \tau/K^p$. Though this procedure can be performed in more than one way, for any fixed choice, the set of segments $\{T^m x\}$ in class (ii) is not empty. Further, if $d(T^m x, \mathcal{O}(y)) \geqslant \tau$, then $\{T^m x\}$ must be a segment in class (ii). Indeed, as one follows the trajectory $\{T^i x\}_{i=0}^{\infty}$, the segments in class (ii) are chosen with a positive frequency, where, as in the previous section, the frequency is defined by $\liminf_{N\to\infty} (N+1)^{-1}(\#\{\text{class (ii) segments in } \{T^i x\}_{i=0}^{N}\})$.

In class (i), for $\epsilon > 0$ sufficiently small and $i = 0, 1, \ldots, p-1$, each function $f_0 + g$ in G achieves a local maximum at $T^{m+i} y$, and $(f_0 + g)(T^{m+i} x) < (f_0 + g)(T^{m+i} y)$. Therefore

$$\sum_{i=0}^{p-1} (f_0 + g)(T^{m+i} x) - p\langle f_0 + g\rangle(y) < 0. \tag{4.1}$$

In class (ii),

$$\begin{aligned}(f_0 + g)(T^m x) - \langle f_0 + g\rangle(y) &= (f_0(T^m x) - 1) + (g(T^m x) - \langle g\rangle(y)) \\ &\leqslant -d(T^m x, \mathcal{O}(y)) + (g(T^m x) - \min_{0\leqslant i<p} g(y)) \leqslant -\frac{\tau}{K^p} + 2\epsilon.\end{aligned} \tag{4.2}$$

If $\epsilon < \tau/(4K^p)$, then $(f_0 + g)(T^m x) < \langle f_0 + g\rangle(y) - \tau/(2K^p)$. Since the segments in class (ii) are chosen with a positive frequency we have

$$\langle f_0 + g\rangle(x) < \langle f_0 + g\rangle(y),$$

a contradiction. □

Remark 4.2. *For C^1 functions, we have a result similar to proposition 4.1, using for f_0 a function that equals $1 - [d(x, \mathcal{O}(y))]^2$ near $\mathcal{O}(y)$. Then of course (4.2) must be modified accordingly, and furthermore the right side of (4.1) must be changed because the points on $\mathcal{O}(y)$ need not be local extrema for $f_0 + g$. In fact, we must use lemma 3.8(b) to show that for every block of consecutive class (i) segments, the sum of the expressions on the left side of (4.1) is bounded above by a constant (independent of ϵ and the length of the block) times ϵ. Then we use the fact that every such block is followed by at least one class (ii) segment. We leave the remaining details to the reader.*

Remark 4.3. *Propositions 3.1 and 4.1 can be proved in a different way, using a type of result that was first introduced by Mañé [5] and later generalized in [8,12]. We state the result here as proposition 4.4 without further comment. Interested readers may refer to these references for details.*

Proposition 4.4. *Let $\{T^i x\}_{i=0}^{\infty}$ be a measure-recurrent optimal orbit for f. Then there exists a Lipschitz function $g_x : \omega(x) \to \boldsymbol{R}$ such that $f(z) \leqslant \langle f\rangle(x) + g_x(Tz) - g_x(z)$, for each $z \in M$; and this inequality becomes an equality for $z \in \omega(x)$.*

Remark 4.5. *By using a similar argument to our proof of proposition 4.1, we can show that, given a Lipschitz function f, if $\{T^i y\}$ is an optimal periodic orbit for f, then we can perturb f to a nearby function $\tilde{f}$ such that $\{T^i y\}$ is optimal for an open set of functions that contains $\tilde{f}$. In other words, the set of functions for which $\{T^i y\}$ is optimal is the closure of an open set.*

In contrast, for optimal non-periodic orbits, we have the following theorem.

Theorem 4.6. *Let $\{T^i x\}_{i=0}^{\infty}$ be a measure-recurrent optimal non-periodic orbit for f. Then there exist arbitrarily small perturbations (within the space of Lipschitz functions) of f under which $\{T^i x\}_{i=0}^{\infty}$ loses optimality.*

Remark 4.7. *Theorem 4.6 implies that optimal non-periodic orbits may lose optimality under small perturbations of f, whereas proposition 4.1 says that the set of functions for which a given periodic orbit is optimal contains an open subset of the Lipschitz functions. In this sense, optimal periodic orbits are more robust than non-periodic ones. This supports conjecture 1.1 in the introduction.*

In the remainder of this section our main goal is to prove theorem 4.6. In developing the proof, we first need some notation. Let $\{T^i x\}$ be a measure-recurrent optimal non-periodic orbit. Given a pair of non-negative integers m and p, we say $S_{m,p} = \{T^m x, T^{m+1} x, \ldots, T^{m+p-1} x\}$ is a *recursive segment* if $\delta_{m,p} := d(T^{m+p} x, T^m x)$ is smaller than $\gamma_{m,p}$, which is the smallest distance between any two points in this segment. Choose c_0, c_1, ϵ_0 and λ and assume $\delta_{m,p}$ is sufficiently small so that lemmas 3.7 and 3.8 and proposition 3.1 hold and that $c_0 \delta_{m,p} < \sigma/2$, where σ is the expansivity constant defined in the proof of proposition 4.1. Then by proposition 3.1(a), $S_{m,p}$ can be $(c_0 \delta_{m,p})$-shadowed by a unique period-p orbit $\{y_{m,p}, T y_{m,p}, \ldots, T^{p-1} y_{m,p}\}$. Let $K > 1$ and $L > 0$ be such that $\|DT\| \leqslant K$ and $\|f\|_{Lip} \leqslant L$.

Now we construct the perturbations that will destroy the optimality of $\{T^i x\}$. Let $y_{m,p}$ be defined as above. Define

$$\tilde{f}_{m,p}(z) = -d(z, \mathcal{O}(y_{m,p})). \tag{4.3}$$

Notice that $\|\tilde{f}_{m,p}\|_{Lip} \leqslant D+1$, where D is the diameter of M. We will prove that given $\epsilon > 0$, $\{T^i x\}$ is not optimal for $f + \epsilon \tilde{f}_{m,p}$ for properly chosen $\tilde{f}_{m,p}$.

Remark 4.8. *If we consider only C^1 perturbations, then the corners of $\tilde{f}_{m,p}$ must be rounded, but we can do so with perturbations that are arbitrarily small in the C^0 norm and do not change its C^1 norm. Thus, once we show that $\{T^i x\}$ is not optimal for $f + \epsilon \tilde{f}_{m,p}$, the same will be true for C^1 approximations of $\tilde{f}_{m,p}$ that are sufficiently close in the C^0 norm.*

Consider the following two classes (whose intersection may be non-empty) of measure-recurrent optimal non-periodic orbits.

Class I. For all $Q > 0$, there exists a recursive segment $S_{m,p}$ such that $\gamma_{m,p}/\delta_{m,p} > Q$.

Class II. The trajectory $\{T^i x\}$ cannot be exponentially approximated by periodic orbits, i.e., for each $\alpha > 0$, there exists $N > 0$ such that $\delta_{m,p} \geqslant \exp(-\alpha p)$, for $0 \leqslant m < \infty$ and $p \geqslant N$.

Notice that if $\omega(x)$ contains a periodic orbit with period p, then $\delta_{m,np}$ can be made arbitrarily small for each $n = 1, 2, 3, \ldots$, and thus $\{T^i x\}$ is not in class II.

The following observation implies that this classification is complete (though possibly overlapping). We remark that the only dynamical assumption used in the proof of this proposition is that $K \geqslant \max\{\|DT\|, 1\}$.

Proposition 4.9. *If $\{T^i x\}$ is not in class I, then it is in class II.*

Proof. Suppose $\{T^i x\}$ is neither in class I nor in class II. Then there exists $Q > 0$, such that $\gamma_{m,p}/\delta_{m,p} \leqslant Q$ for every recursive segment $S_{m,p}$. Also, there exists $\alpha > 0$ and arbitrarily long recursive segments $S_{m,p}$ such that $\delta_{m,p} < \exp(-\alpha p)$. We now prove that this will lead to a contradiction.

Let S_{m_0,p_0} be a recursive segment with $\delta_{m_0,p_0} < \exp(-\alpha p_0)$. Since $\{T^i x\}$ is not in class I, there exists a recursive segment S_{m_1,p_1} contained in S_{m_0,p_0} (with $m_{i+1} + p_{i+1} < m_i + p_i$) such that $\delta_{m_1,p_1} \leqslant Q\delta_{m_0,p_0}$. Repeat this procedure, starting with S_{m_i,p_i} at each time, until we finally get a segment that contains only two elements, where the length cannot be further reduced. Thus, we have constructed a sequence of recursive segments $\{S_{m_i,p_i}\}_{i=0}^n$ with the properties that

$$\delta_{m_{i+1},p_{i+1}} \leqslant Q\delta_{m_i,p_i}, \tag{4.4}$$

and the last recursive segment S_{m_n,p_n} in this sequence contains exactly two elements, i.e. $p_n = 2$. Let

$$\tau = \inf_{0<i-j<2\ln Q/\alpha} d(T^i x, T^j x). \tag{4.5}$$

Notice that by a similar construction as above, starting with a recursive segment of length at most $2\ln Q/\alpha$ and distance at most 2τ, there exists k such that $d(T^k x, T^{k+1}x) \leqslant 2\tau Q^{2\ln Q/\alpha}$, and therefore $\tau > 0$.

We first show that $p_i - p_{i+1}$, the length that is reduced in the ith step of the previous procedure, is large when δ_{m_i,p_i} is small.

$$\begin{aligned} d(T^{m_{i+1}+p_i}x, T^{m_{i+1}+p_{i+1}}x) &\leqslant d(T^{m_{i+1}+p_i}x, T^{m_{i+1}}x) + d(T^{m_{i+1}+p_{i+1}}x, T^{m_{i+1}}x) \\ &\leqslant K^{m_{i+1}-m_i} d(T^{m_i+p_i}x, T^{m_i}x) + d(T^{m_{i+1}+p_{i+1}}x, T^{m_{i+1}}x) \\ &= K^{m_{i+1}-m_i}\delta_{m_i,p_i} + \delta_{m_{i+1},p_{i+1}} \\ &\leqslant (K^{m_{i+1}-m_i} + Q)\delta_{m_i,p_i}, \end{aligned} \tag{4.6}$$

where the last step follows from (4.4).

Since $m_i + p_i > m_{i+1} + p_{i+1}$, we have $m_{i+1} - m_i < p_i - p_{i+1}$. Thus (4.6) gives

$$d(T^{m_{i+1}+p_i}x, T^{m_{i+1}+p_{i+1}}x) < (K^{p_i-p_{i+1}} + Q)\delta_{m_i,p_i}. \tag{4.7}$$

Let $\delta^* = (K^{2\ln Q/\alpha} + Q)^{-1}\tau > 0$, where τ is defined in equation (4.5). Then (4.7) implies for $\delta_{m_i,p_i} \leqslant \delta^*$, we have

$$p_i - p_{i+1} > 2\ln Q/\alpha. \tag{4.8}$$

By hypothesis, S_{m_0,p_0} can be chosen with p_0 arbitrarily large. Choose p_0 to be sufficiently large such that $\delta_{m_0,p_0} < (\delta^*)^2$. Define

$$n^* = \left[\frac{\ln(\delta^*/\delta_{m_0,p_0})}{\ln Q}\right], \tag{4.9}$$

where $[\,\cdot\,]$ denotes taking the integer part. For $0 \leqslant i \leqslant n^*$, we have

$$\delta_{m_i,p_i} \leqslant Q^i\delta_{m_0,p_0} \leqslant Q^{\frac{\ln(\delta^*/\delta_{m_0,p_0})}{\ln Q}} \cdot \delta_{m_0,p_0} = \delta^*.$$

Thus (4.8) implies $p_i - p_{i+1} > 2\ln Q/\alpha$. Hence,

$$\begin{aligned} p_0 > \sum_{i=0}^{n^*}(p_i - p_{i+1}) > (n^*+1)\cdot\frac{2\ln Q}{\alpha} &> \left(\frac{\ln\delta^* - \ln\delta_{m_0,p_0}}{\ln Q}\right)\cdot\frac{2\ln Q}{\alpha} \\ &= -\frac{2}{\alpha}(\ln\delta_{m_0,p_0} - \ln\delta^*) > -\frac{\ln\delta_{m_0,p_0}}{\alpha}. \end{aligned} \tag{4.10}$$

Inequality (4.10) yields $\delta_{m_0,p_0} > \exp(-\alpha p_0)$, a contradiction. □

Theorem 4.6 then follows from the following two lemmas.

Lemma 4.10. *If $\{T^i x\}$ is in class I, then for every $\epsilon > 0$, there exists a periodic point $y_{m,p}$ such that $\langle f + \epsilon \tilde{f}_{m,p}\rangle(x) < \langle f + \epsilon \tilde{f}_{m,p}\rangle(y_{m,p})$ (where $y_{m,p}$ and $\tilde{f}_{m,p}$ are defined earlier in this section, after recursive segments). In particular, $\{T^i x\}$ is non-optimal for $f + \epsilon \tilde{f}_{m,p}$.*

Lemma 4.11. *The conclusion of lemma 4.10 also holds if $\{T^i x\}$ is in class II.*

Proof of lemma 4.10. If there exists a periodic orbit in $\omega(x)$, then by applying corollary 3.2 and proposition 4.1, we are done. Thus, we need only to consider the case where there are no periodic orbits in $\omega(x)$.

Since $\{T^i x\}$ is in class I, for given $\rho > 8c_0$, we can choose a recursive segment $S_{m,p} := \{T^i x\}_{i=m}^{m+p-1}$ such that $\gamma_{m,p}/\delta_{m,p} \geqslant \rho$. Further,we can choose $\delta_{m,p}$ small enough that $S_{m,p}$ is $(c_0\delta_{m,p})$-shadowed by the period-p orbit $\{T^i y_{m,p}\}_{i=0}^{p-1}$. Let $y = y_{m,p}$. Notice that the smallest separation between any two points in $\mathcal{O}(y)$ is bounded from below by $\gamma_{m,p} - 2c_0\delta_{m,p} > 3\gamma_{m,p}/4$. Since M is compact, by making p sufficiently large, we can make $\gamma_{m,p} < \sigma$ for all m, where σ is an expansivity constant as defined in the proof of proposition 4.1. Let

$$\delta_0 = \inf_{i,j} d(T^i x, T^j y) > 0. \tag{4.11}$$

Fixing y (and hence p), we now change our selection of $S_{m,p}$, if necessary, so that $\delta_{m,p} < 2K^p\delta_0$. To avoid confusion, we use $\delta_{m,p}^{old}$ and $\gamma_{m,p}^{old}$ to denote the corresponding terms for the initially selected $S_{m,p}$. We have $\delta_{m,p} \leqslant \delta_{m,p}^{old}$ and

$$\gamma_{m,p} > \frac{3\gamma_{m,p}^{old}}{4} - 2c_0\delta_{m,p} > \frac{\gamma_{m,p}^{old}}{2},$$

hence $\gamma_{m,p}/\delta_{m,p} \geqslant \gamma_{m,p}^{old}/(2\delta_{m,p}^{old}) \geqslant \rho/2$. Therefore, the recursive segment $S_{m,p}$ selected in this way can have arbitrarily large ratio $\gamma_{m,p}/\delta_{m,p}$.

From proposition 3.1(b),

$$\langle f\rangle(y_{m,p}) > \langle f\rangle(x) - \frac{c_0c_1\delta_{m,p}}{p}. \tag{4.12}$$

Using a technique similar to the proof of proposition 4.1, we divide $\{T^i x\}$ into infinitely many segments such that each segment $\{T^k x, \ldots, T^{k+\ell}x\}$ has the following properties: (a) there exists $j > 0$ such that $d(T^{k+i}x, T^{j+i}y) \leqslant \gamma_{m,p}/(8K)$, for $0 \leqslant i < \ell$; and (b) $d(T^{k+\ell}x, \mathcal{O}(y)) > \gamma_{m,p}/(8K)$. (Some segments will have $\ell = 0$.) To see that $\{T^i x\}$ can be divided in this way, we notice that $\gamma_{m,p} < \sigma$. Hence if $d(T^k x, T^j y) \leqslant \gamma_{m,p}/(8K)$, then there exists a smallest ℓ such that $d(T^{k+\ell}x, T^{j+\ell}y) > \gamma_{m,p}/(8K)$. Since $d(T^{k+\ell}x, T^{j+\ell}y) \leqslant \gamma_{m,p}/8$, and the smallest separation between any two points in $\mathcal{O}(y)$ is bounded from below by $3\gamma_{m,p}/4$, we have

$$d(T^{k+\ell}x, \mathcal{O}(y)) = d(T^{k+\ell}x, T^{j+\ell}y) > \gamma_{m,p}/(8K).$$

From lemma 3.8(b),

$$d(T^{k+i}x, T^{j+i}y) \leqslant c_0\lambda^{\min(i,\ell-1-i)}\gamma_{m,p}, \qquad \text{for} \quad 0 \leqslant i < \ell. \tag{4.13}$$

From (4.13), $\delta_0 \leqslant c_0\lambda^{-1+\ell/2}\gamma_{m,p}$, where δ_0 is defined in equation (4.11), so

$$\ell \leqslant \frac{2\ln(c_0\gamma_{m,p}/\delta_0)}{\ln(1/\lambda)} + 2. \tag{4.14}$$

Recall the definition (4.3) of $\tilde{f}_{m,p}$. Since $\delta_{m,p} < 2K^p\delta_0$,

$$\langle \tilde{f}_{m,p}\rangle(y_{m,p}) - \frac{1}{\ell+1}\sum_{i=0}^{\ell}\tilde{f}_{m,p}(T^{k+i}x) = \frac{1}{\ell+1}\sum_{i=0}^{\ell} d(T^{k+i}x, \mathcal{O}(y)) > \frac{1}{\ell+1}\cdot\frac{\gamma_{m,p}}{8K}$$
$$\geqslant \frac{\ln(1/\lambda)}{2\ln(c_0\gamma_{m,p}/\delta_0) + 3\ln(1/\lambda)}\cdot\frac{\gamma_{m,p}}{8K}$$
$$> \frac{\ln(1/\lambda)}{2\ln(2c_0K^p\gamma_{m,p}/\delta_{m,p}) + 3\ln(1/\lambda)}\cdot\frac{\gamma_{m,p}}{8K}. \qquad (4.15)$$

Since (4.15) is true for every segment $\{T^kx, \ldots, T^{k+\ell}x\}$,

$$\langle \tilde{f}_{m,p}\rangle(y_{m,p}) - \langle \tilde{f}_{m,p}\rangle(x) \geqslant \frac{\ln(1/\lambda)}{2\ln(2c_0\gamma_{m,p}/\delta_{m,p}) + 2p\ln K + 3\ln(1/\lambda)}\cdot\frac{\gamma_{m,p}}{8K}. \qquad (4.16)$$

Given $\epsilon > 0$, for sufficiently large $\gamma_{m,p}/\delta_{m,p}$, we have

$$\frac{\gamma_{m,p}}{\delta_{m,p}} > \frac{8c_0c_1K[2\ln(2c_0\gamma_{m,p}/\delta_{m,p}) + 2p\ln K + 3\ln(1/\lambda)]}{\epsilon p\ln(1/\lambda)}.$$

Therefore,

$$\frac{c_0c_1\delta_{m,p}}{p} < \frac{\epsilon\ln(1/\lambda)}{2\ln(2c_0\gamma_{m,p}/\delta_{m,p}) + 2p\ln K + 3\ln(1/\lambda)}\cdot\frac{\gamma_{m,p}}{8K}. \qquad (4.17)$$

Combining (4.16) and (4.17), together with (4.12), yields $\langle f + \epsilon\tilde{f}_{m,p}\rangle(x) < \langle f + \epsilon\tilde{f}_{m,p}\rangle(y_{m,p})$. □

Remark 4.12. *Using more involved arguments, we can prove for $\{T^ix\}$ in class I that for all $\epsilon > 0$, there exist $y_{m,p}$ and $\tilde{f}_{m,p}$ such that $\langle f + \epsilon\tilde{f}_{m,p}\rangle(z) < \langle f + \epsilon\tilde{f}_{m,p}\rangle(y_{m,p})$, for all measure-recurrent orbits $\{T^iz\}$ that are different from $\{T^iy_{m,p}\}$; so, $\{T^iy_{m,p}\}$ is an optimal periodic orbit for $f + \epsilon\tilde{f}_{m,p}$.*

Though we believe that there exist orbits in class II, it is not clear to us that there are any orbits not in class I. If all orbits are in class I, then remarks 4.5 and 4.12 imply that the set of functions with optimal periodic orbits contains an open and dense subset of the Lipschitz functions. By remarks 4.2 and 4.8, the same would then be true for the space of C^1 functions. Thus, we claim that conjecture 1.1 follows from the following conjecture.

Conjecture 4.13. *For axiom A or uniformly expanding T, all orbits of T are in class I.*

The following lemma will be used in the proof of lemma 4.11.

Lemma 4.14. *There exists $N_0 > 0$, such that for any recursive segment $S_{m,p}$ with $p > 2N_0$, the following properties hold.*

(a) $d(T^iy_{m,p}, T^jy_{m,p}) \geqslant \delta_{m,p}/2$, *for* $N_0 \leqslant i < j \leqslant p - N_0$.
(b) For every segment $\{T^{j+i}x\}_{i=0}^{s}$ that stays in the $\delta_{m,p}/(4K)$-neighbourhood of $\{T^{N_0}y, T^{N_0+1}, \ldots, T^{p-N_0}y\}$, there exists k between N_0 and $p - N_0$ such that $d(T^{j+i}x, T^{k+i}y) < \delta_{m,p}/(4K)$ for $0 \leqslant i \leqslant s$. (Recall that K is an upper bound on $\|DT\|$.)

Proof. Let us first prove part (a). Since $S_{m,p}$ is a recursive segment, we have $d(T^{m+i}x, T^{m+j}x) \geqslant \delta_{m,p}$ for $0 \leqslant i < j \leqslant p$. Lemma 3.8(a) implies

$$d(T^{m+i}x, T^iy_{m,p}) < c_0\lambda^{\min(i,p-i)}\delta_{m,p}, \qquad \text{for} \quad 0 \leqslant i \leqslant p.$$

Let $N_0 = [\ln(4c_0)/\ln(1/\lambda)] + 1$, where $[\cdot]$ denotes the integer part of a number. Then for $N_0 \leqslant i \leqslant p - N_0$, we have $d(T^{m+i}x, T^i y_{m,p}) < c_0\lambda^{N_0}\delta_{m,p} \leqslant \delta_{m,p}/4$. Thus, if $N_0 \leqslant i < j \leqslant p - N_0$, then

$$d(T^i y_{m,p}, T^j y_{m,p}) \geqslant d(T^{m+i}x, T^{m+j}x) - d(T^i y_{m,p}, T^{m+i}x)$$
$$-d(T^j y_{m,p}, T^{m+j}x) \geqslant \frac{\delta_{m,p}}{2}.$$

Next we prove part (*b*). Notice that there exists a unique k between N_0 and $p - N_0$ such that $d(T^j x, T^k y) < \delta_{m,p}/(4K)$. It follows that $d(T^{j+1}x, T^{k+1}y) < \delta_{m,p}/4$. From (*a*), $d(T^{j+1}x, T^{k+1}y) = \min_{N_0<i\leqslant p-N_0} d(T^{j+1}x, T^i y)$. Therefore from our assumption, $d(T^{j+1}x, T^{k+1}y) < \delta_{m,p}/(4K)$. By repeating the previous procedure, we prove that $d(T^{j+i}x, T^{k+i}y) < \delta_{m,p}/(4K)$, for $0 \leqslant i \leqslant s$. □

Proof of lemma 4.11. Suppose there exists $\epsilon > 0$ such that for every $S_{m,p}$, $\langle f + \epsilon \tilde{f}_{m,p}\rangle(x) \geqslant \langle f + \epsilon \tilde{f}_{m,p}\rangle(y_{m,p})$. We will prove that $\{T^i x\}_{i=0}^{\infty}$ is not in class II, i.e., it can be exponentially approximated by periodic orbits, a contradiction.

Let μ_x be the measure generated by x. Then

$$\langle \tilde{f}_{m,p}\rangle(y_{m,p}) - \langle \tilde{f}_{m,p}\rangle(x) = \int d(z, \mathcal{O}(y_{m,p}))\mu_x(\mathrm{d}z). \tag{4.18}$$

From proposition 3.1(b),

$$\langle f\rangle(y_{m,p}) - \langle f\rangle(x) > -\frac{c_0 c_1 \delta_{m,p}}{p}. \tag{4.19}$$

Since $\langle f + \epsilon \tilde{f}_{m,p}\rangle(y_{m,p}) \leqslant \langle f + \epsilon \tilde{f}_{m,p}\rangle(x)$, (4.18) and (4.19) yield

$$\int d(z, \mathcal{O}(y_{m,p}))\mu_x(\mathrm{d}z) < \frac{c_0 c_1 \delta_{m,p}}{\epsilon p}. \tag{4.20}$$

Assume $\epsilon < c_0 c_1 K$; if not, replace ϵ by a smaller value. Let S_{m_0,p_0} be a recursive segment such that $p_0 > 100c_0c_1KN_0/\epsilon$, where N_0 is as in lemma 4.14, $\delta_{m_0,p_0} < 4K\epsilon_0$, where ϵ_0 is as in lemma 3.8, and $\delta_{m,p} > \delta_{m_0,p_0}$ for all $p < p_0$ and $m \geqslant 0$. Let $n = [\epsilon p_0/(100c_0c_1KN_0)] \geqslant 1$, where $[\cdot]$ again denotes the integer part of a number. Notice that $12nN_0 < p_0$ and $n > \epsilon p_0/(200c_0c_1KN_0)$. We divide $\{T^i x\}$ into segments of length $12nN_0$ such that each segment has one and only one of the following properties: (*a*) it contains only points in the $\delta_{m_0,p_0}/(4K)$-neighbourhood of $\mathcal{O}(y_{m_0,p_0})$—we say such a segment is an inner segment; (*b*) it contains at least one point that is not in the $\delta_{m_0,p_0}/(4K)$-neighbourhood of $\mathcal{O}(y_{m_0,p_0})$ – we say such a segment is an outer segment. Let $y = y_{m_0,p_0}$, and $\Delta = \{T^{p-N_0+1}y, T^{p-N_0+2}y, \ldots, T^p y = y, Ty, T^2y, \ldots, T^{N_0-1}y\}$. We claim that there cannot exist two points in the same inner segment that are both in the $\delta_{m_0,p_0}/(4K)$-neighbourhood of the same point in Δ; otherwise there exists a segment $S_{m,p}$ with $\delta_{m,p} < \delta_{m_0,p_0}/(2K) < \delta_{m_0,p_0}$ and $p < 12nN_0 < p_0$, which is a contradiction. Thus, in each inner segment there are at most $2N_0$ points in the $\delta_{m_0,p_0}/(4K)$-neighbourhood of Δ. After removing $2N_0$ points from a segment of length $12nN_0 \geqslant 4n(2N_0 + 1)$, a continuous segment of length at least $4n$ must remain. Therefore, each inner segment contains a subsegment $\{T^{j'+i}x\}_{i=0}^{4n-1}$ that stays in the $\delta_{m_0,p_0}/(4K)$-neighbourhood of $\mathcal{O}(y) \setminus \Delta$. Lemma 4.14 implies that this subsegment $\delta_{m_0,p_0}/(4K)$-shadows a segment of $\mathcal{O}(y)$. From lemma 3.8(b),

$$d(T^{j'+i}x, \mathcal{O}(y_{m_0,p_0})) \leqslant 2c_0\lambda^{\min(i,4n-1-i)}\delta_{m_0,p_0}, \qquad \text{for} \quad 0 \leqslant i < 4n. \tag{4.21}$$

In particular, if $n \leqslant i < 3n$, then (4.21) implies

$$d(T^{j'+i}x, \mathcal{O}(y_{m_0,p_0})) \leqslant 2c_0\lambda^n\delta_{m_0,p_0} < 2c_0\lambda^{\epsilon p_0/(200c_0c_1KN_0)}\delta_{m_0,p_0}. \tag{4.22}$$

Let $\alpha = \epsilon \ln(1/\lambda)/(8000c_0c_1KN_0^2)$. Then from (4.22) we have

$$d(T^{j'+i}x, \mathcal{O}(y_{m_0,p_0})) < 2c_0\mathrm{e}^{-40\alpha N_0 p_0}\delta_{m_0,p_0}. \qquad (4.23)$$

Thus, each inner segment contains at least $2n$ points that lie within $2c_0\exp(-40\alpha N_0 p_0)\delta_{m_0,p_0}$ of $\mathcal{O}(y_{m_0,p_0})$.

We claim that there exists k such that $I_k := \{T^i x\}_{i=k+1}^{k+30N_0p_0}$ contains at least $100c_0c_1KN_0/\epsilon$ inner segments, and thus contains at least $200c_0c_1KnN_0/\epsilon$ points in the $2c_0\exp(-40\alpha N_0p_0)\delta_{m_0,p_0}$-neighbourhood of $\mathcal{O}(y_{m_0,p_0})$. To prove this claim, we first observe that from (4.20) there exists k, such that

$$\frac{1}{30N_0p_0}\sum_{i=k+1}^{k+30N_0p_0} d(T^i x, \mathcal{O}(y_{m_0,p_0})) < \frac{c_0c_1\delta_{m_0,p_0}}{\epsilon p_0}.$$

The average of $d(T^i x, \mathcal{O}(y_{m_0,p_0}))$ over each outer segment is larger than $\delta_{m_0,p_0}/(48KnN_0) > 2c_0c_1\delta_{m_0,p_0}/(\epsilon p_0)$ (recall that $n = [\epsilon p_0/(100c_0c_1KN_0)]$). Thus, at least half of the total points in I_k are contained in either an inner segment or a partial segment at the beginning or end of I_k. We conclude that there must be at least $15N_0p_0/(12nN_0) - 2 > 125c_0c_1KN_0/\epsilon - 2 > 100c_0c_1KN_0/\epsilon$ inner segments.

Since $200c_0c_1KnN_0/\epsilon > p_0$, there are at least two points, $T^{m_1}x$ and $T^{m_1+p_1}x$ with $p_1 < 30N_0p_0$, within $2c_0\exp(-40\alpha N_0p_0)\delta_{m_0,p_0}$ of the same point in $\mathcal{O}(y_{m_0,p_0})$; hence their distance is less than $4c_0\mathrm{e}^{-40\alpha N_0p_0}\delta_{m_0,p_0}$. We can choose p_0 as large as we want; assume then that $4c_0\mathrm{e}^{-10\alpha N_0p_0} < 1$. Moreover, we can choose S_{m_1,p_1} to be a recursive segment. Then $\delta_{m_1,p_1} < \mathrm{e}^{-30\alpha N_0p_0}\delta_{m_0,p_0} < \mathrm{e}^{-\alpha p_1}\delta_{m_0,p_0}$.

Repeat the previous arguments to construct a sequence of recursive segments $\{S_{m_i,p_i}\}_{i=0}^{\infty}$ for which $\delta_{m_i,p_i} < \mathrm{e}^{-\alpha p_i}\delta_{m_{i-1},p_{i-1}} < \exp(-\alpha\sum_{j=1}^{i} p_j)\delta_{m_0,p_0}$. In particular, $\delta_{m_i,p_i} \to 0$ as $i \to \infty$. By our assumption that $\{T^i x\}$ is in class II, $\omega(x)$ contains no periodic orbits, and thus we must have $\lim_{i\to\infty} p_i = \infty$. On the other hand, we can choose $\alpha' \in (0,\alpha)$ such that $\delta_{m_i,p_i} < \mathrm{e}^{-\alpha p_i}\delta_{m_0,p_0} < \mathrm{e}^{-\alpha' p_i}$, which contradicts the assumption that $\{T^i x\}$ is in class II. □

5. Summary

We have discussed some basic properties of optimal orbits (in the sense of definition 2.1). Assuming that the dynamical system is given by either an axiom A diffeomorphism or a uniformly expanding non-invertible map, our main questions are the plausibility of conjecture 1.1, which was inspired by numerical results and heuristic arguments [1, 2], and question 1.2.

In recent years, the idea that periodic orbits act as the skeleton (see [15], for example) of dynamical systems has become popular, and people have been using such ideas in computing certain average quantities (such as entropy and Lyapunov exponents) in dynamical systems. The investigation of periodic orbits as optimal orbits should offer new insights into the important role of periodic orbits in dynamical systems. In this paper, we try to study this problem from a mathematical point of view.

Our main discussion about conjecture 1.1 is presented in section 4. Let $\mathcal{F}_x$ be the set of Lipschitz functions f such that $\mathcal{O}(x)$ is optimal. In proposition 4.1, we prove that if x is periodic, then $\mathcal{F}_x$ contains an open subset; on the other hand, if x generates a non-periodic invariant measure, then theorem 4.6 implies that $\mathcal{F}_x$ contains no open subset. Let $\mathcal{F}_{per} = \bigcup\{\mathcal{F}_x : x \text{ is periodic}\}$. Notice that conjecture 1.1 can be equivalently restated as follows: for a given T, $\mathcal{F}_{per}$ contains an open and dense subset. In proposition 4.1, we prove that $\mathcal{F}_{per}$ contains an open subset. Furthermore, in theorem 4.6, we prove that each non-periodic optimal orbit loses optimality under certain small perturbations of f. Remarks 4.2 and 4.8 show how to apply these results to the space of C^1 functions. Hence, to complete the proof of conjecture 1.1, it remains to be proved (based on remark 4.5) that the perturbations constructed in the proof of theorem 4.6, or some other small perturbations, make a periodic orbit optimal. As we stated in remark 4.12, the proof of lemma 4.10 can be extended to prove for class I orbits that the previous statement is true.

Although we divide orbits into two classes, these classes may overlap, and indeed class I may contain all orbits. Though we believe that it is possible, with some effort, to construct for the map $x \mapsto 2x (\mathrm{mod}\ 1)$ an orbit that is in class II, consideration of this map, which can be done entirely through symbolic dynamics, makes us doubt that an orbit outside class I exists. This leads us to formulate conjecture 4.13, which we think is an interesting question in its own right, especially in the case of $x \mapsto 2x (\mathrm{mod}\ 1)$, where it can be formulated as a combinatorial problem on symbol sequences. A positive answer to this conjecture for any class of hyperbolic maps would allow us to complete the proof of conjecture 1.1 for that class. We emphasize though, that we believe conjecture 1.1 holds regardless of whether conjecture 4.13 does.

We gave a preliminary answer to question 1.2 in corollary 3.4, in the sense that we gave a bound on the rate at which the maximum average of a given performance function f over periodic orbits up to a given period p converges to the optimal average as p increases. Specifically, we showed that if d_p is the difference between these two averages, then $d_p \leqslant C p^{-1/m}$, where m is the dimension of the ambient manifold. As we suggested in remark 3.5, the bound could probably be improved significantly by getting a better idea of the 'worst case' metric recurrence properties of an arbitrary trajectory $\{T^i x\}$, in the following sense. Define, as in remark 3.5, ϵ_p to be the closest recurrence in the trajectory within p iterations; then $d_p \leqslant C\epsilon_p$. How quickly must ϵ_p approach 0 as p increases? Again, we think this is an interesting question in its own right, and what the 'worst case' trajectory is from this point of view seems very unclear even for $x \mapsto 2x (\mathrm{mod}\ 1)$; again for this map the problem can be considered combinatorially in terms of symbol sequences.

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