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Collapsing of chaos in one dimensional maps

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Abstract

In their numerical investigation of the family of one dimensional maps $f_\ell(x) = 1 - 2|x|^\ell$, where $\ell > 2$, Diamond et al. [P. Diamond et al., Physica D 86 (1999) 559–571] have observed the surprising numerical phenomenon that a large fraction of initial conditions chosen at random eventually wind up at -1, a repelling fixed point. This is a numerical artifact because the continuous maps are chaotic and almost every (true) trajectory can be shown to be dense in $[-1, 1]$. The goal of this paper is to extend and resolve this obvious contradiction. We model the numerical simulation with a randomly selected map. While they used 27 bit precision in computing f_ℓ, we prove for our model that this numerical artifact persists for an arbitrary high numerical prevision. The fraction of initial points eventually winding up at -1 remains bounded away from 0 for every numerical precision. ©2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Because of the prevalence of numerical calculations of dynamical systems, it is important to identify special situations in which numerical simulations yield strikingly incorrect results. Computer simulation of chaotic dynamical systems can be very different from the real systems. Recently, a rather surprising numerical artifact was identified by Diamond et al. in two nice papers [1,2]. For the family of maps

$$f_\ell(x) = 1 - 2|x|^\ell, \quad x \in [-1, 1], \quad \ell > 2, \tag{1.1}$$

f_ℓ is chaotic, as we will show in the next section. In particular, almost every trajectory is dense in the whole interval. Diamond et al. numerically investigated trajectories of (1.1) and reported that for an initial point chosen at random

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in $(-1, 1)$, the trajectory has a considerable probability of reaching the point -1. Note that -1 is a repelling fixed point, so the trajectory remains at -1 thereafter.

Indeed, numerically every trajectory in $[-1, 1]$ is eventually periodic, since there are only finitely many numbers available to the computer. If a (numerical) trajectory reaches an unstable fixed point (which is -1 in this case), we say the trajectory *collapses*. Note that the numerical trajectory will stay at -1 thereafter. Diamond et al. found that for an initial point chosen at random in $(-1, 1)$, the probability of collapse is roughly 70% for ℓ approximately 3. This probability fluctuates wildly as ℓ varies, and 70% is an average for ℓ near 3. They use a fixed point representation of precision $\epsilon = 2^{-27}$. Apparently, they compute $f_\ell(x)$ accurately and then round to the nearest integer of the form $k \cdot 2^{-27}$, where k is an integer. Hence -1 remains a numerical fixed point.

The aim of this paper is to present a computational model for which we can prove that this artifact persists even if numerical precision is increased arbitrarily.

Indeed, there is a trivial example where collapsing occurs. Let T be the tent map on $[0, 1]$ with slope 2. Then T is clearly chaotic and preserves Lebesgue measure on $[0, 1]$. On the other hand, if numbers have binary representations, it is easy to see that every numerical trajectory will eventually become 0 (and will of course stay at 0). However, for tent maps collapsing is unusual in the sense that it disappears when the slope is slightly less than 2; whereas for f_ℓ, where $\ell > 2$, the probability is positive.

For high precision calculations, the time to first repeat will be large, and it seems natural to assume that before a numerical trajectory repeats, it has the same statistics as those of a typical true trajectory. This assumption leads us to model a numerical simulation of f_ℓ by a mapping that is randomly selected from the collection of all mappings defined on a finite set. Similar ideas have also been discussed in several other papers, see, e.g. [1–7].

In developing the model, we need definitions. Given a map f, we defined the *fraction* of the iterates of the orbit $(f^n(x))_{i=1}^{\infty}$ lying in a set S by

$$F(x, S) := \lim_{n\to\infty} \frac{\#\{f^i(x) \in S : 1 \le i \le n\}}{n}.$$

Write $N(r, S) := \{x : \mathrm{dist}(x, S) \le r\}$. The *natural measure generated by the map* f is defined by

$$\mu_f(S) := \lim_{r\to 0} F(x, N(r, S))$$

for each closed set S, as long as all x except for a measure zero set gives the same answer.

In the next section we will prove that f_ℓ, $\ell > 2$, the natural measure exists and is absolutely continuous with respect to the Lebesgue measure, and that the support of the measure is $[-1, 1]$. For the moment we assume the existence of natural measure and start describing our model. Let μ be the natural measure.

A computer using fixed precision will have some number N of points (equally spaced) from -1 to $+1$. We will investigate the set of maps on these points subject to the condition that -1 is a fixed point and $+1$ maps to -1 and as described below the maps have a probability distribution on them that is determined by the reduced μ. We partition the interval $[-1, 1]$ with the uniform grid $\Delta_N = \{x_0, x_1, \dots, x_{2N}\}$, where $x_k = -1 + k/N$. Let $\tilde{p}_k$ be the measure (with respect to μ) of the interval of length $1/N$ centered at x_k.

The "induced" probability distribution on Δ_N is defined as follows:

$$p_k := \frac{\tilde{p}_k}{1 - \tilde{p}_0}, \quad \text{for } 0 < k \le 2N. \tag{1.2}$$

Notice that $\sum_{k=1}^{2N} p_k = 1$. Hence Eq. (1.2) gives a probability distribution.

Let $\mathcal{T}_{N,\ell}$ be the collection of $T : \Delta_N \to \Delta_N$ such that $T(-1) = -1$ and $T(+1) = -1$. We choose a map T from $\mathcal{T}_{N,\ell}$ (ℓ determines the probability distribution) at random according to the following:

$$P\{T(x_i) = x_j\} = p_j, \quad \text{for} \quad 0 < i < 2N \quad \text{and} \quad 0 < j \leq 2N. \tag{1.3}$$

Eq. (1.3) gives a probability depending on ℓ.

This model is motivated by fixed precision arithmetic. For floating point computation, the floating numbers are denser at 0 than anywhere else, and this does not significantly affect computational results for this problem.

Theorem 1.1. *Let $\ell > 2$. Let x be an initial point chosen randomly from Δ_N with uniform distribution, and let T be a mapping chosen randomly from $\mathcal{T}_{N,\ell}$ accordingly to Eq.* (1.3). *Write $\mathcal{P}_{collapse}$ for the probability that there exists an n such that $T^n(x) = -1$. Then $\mathcal{P}_{collapse}$ depends only on N and ℓ and*

$$\liminf_{N\to\infty} \mathcal{P}_{collapse} > 0.$$

The distribution of initial points is not important as long as it is induced by a probability measure that is an equivalent Lebesgue measure. Thus to simplify our proof of Theorem 1.1 we are able to consider the initial distribution to be given by Eq. (1.2) instead.

When applied to numerical simulation, $\liminf_{N\to\infty}\mathcal{P}_{collapse}$ stands for a lower bound of the probability with which an initial condition chosen at random eventually maps to the fixed point -1. Theorem 1.1 means that for $\ell > 2$ this probability is bounded away from zero.

Diamond et al. also use randomly selected mappings in their analysis, but they use a different and simpler statistical model (see also [8] for their recent discussion), which assumes the distribution on Δ_N is uniform except at 1, and the weight at 1 is adjusted to yield the agreement between $\mathcal{P}_{collapse}$ obtained from theoretical derivation and numerical simulation. The advantage of our model is that it is based on the actual properties of the invariant density (which we establish here). Furthermore, we can estimate how $\mathcal{P}_{collapse}$ depends on ℓ (Eq. (3.23)). In particular,

$$1, \quad \mathcal{P}^{lower}_{collapse}(l = 2 + \delta) \sim \delta^{1/2}, \tag{1.4}$$

$$2, \quad \mathcal{P}^{lower}_{collapse}(l \to \infty) = 1, \tag{1.5}$$

where $\mathcal{P}^{lower}_{collapse}$ is our lower bound for $\mathcal{P}_{collapse}$ computed from Eq. (3.23). The quantity $\mathcal{P}_{collapse}$ can be used as an a priori estimate of how much statistical information is distorted in numerical simulations.

2. Invariant measure

Despite all the numerical evidence that many chaotic systems have a natural measure, its existence can be mathematically justified only for a few special cases [9–16]. Fortunately, when $\ell \geq 2$ the map f_ℓ belongs to such cases. The goal of this section is to verify the existence of the natural measure and analyze this measure quantitatively. Indeed, we will prove the following proposition.

Proposition 2.1. *For f_ℓ, $\ell > 2$, there exists a unique invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure. The density $\rho(x)$ for this measure is continous on $(-1, 1)$ and bounded away from 0. For each ℓ; the following limits exist:*

$$\alpha_1 := \lim_{x\to 1} \rho(x)(1-x)^{1-1/\ell}, \tag{2.1}$$

$$\alpha_0 := \lim_{x\to -1} \rho(x)(1+x)^{1-1/\ell} \tag{2.2}$$

and they satisfy

$$\alpha_0 := \alpha_1/((2\ell)^{1/\ell} - 1). \tag{2.3}$$

To prove Proposition 2.1, we first observe that its Schwarzian derivative Sf_ℓ is negative, where

$$Sf_\ell(x) := \frac{f_\ell'''(x)}{f_\ell'(x)} - \frac{3}{2}\left(\frac{f_\ell'''(x)}{f_\ell'(x)}\right)^2.$$

Next, since $f_\ell^2(0) = -1$, and -1 is an unstable fixed point, $|(f_\ell^n)'(f_\ell(c))|$ grows exponentially in n. From the following proposition, f_ℓ has a natural measure μ and μ is absolutely continuous with respect to Lebesgue measure. Let ρ be its density.

Proposition 2.2 (see [17], Theorem V 4.1). *Let $f : [-1, 1] \to [-1, 1]$ be a unimodal* (*i.e. f has only one critical point*) *C^3 map with negative Schwarzian derivative and assume that the critical point c of f is of finite order $\ell \geq 1$, i.e., assume that there are constants O_1, O_2 such that*

$$O_1|x-c|^{\ell-1} \leq |f'(x)| \leq O_2|x-c|^{\ell-1}.$$

Furthermore, assume that the growth-rate of $|(f^n)'(f(c))|$ is so fast that

$$\sum_{n=1}^{\infty}|(f_n)'(f(c))|^{-1/\ell} < \infty. \tag{2.4}$$

Then f has a unique invariant probability measure μ that is absolutely continuous with respect to Lebesgue measure. Furthermore, there exists a constant G such that

$$\mu(A) \leq G|A|^{1/\ell},$$

for any measurable set $A \subset (-1, 1)$ where $|A|$ is the Lebesgue measure of A.

Note that f_ℓ satisfies Eq. (2.4) since $(f^n)'(f_\ell(c))$ grows exponentially as $n \to \infty$. We need in addition to show that the invariant density ρ is continuous. To this end we first define a first return map R_ℓ. Note that $f_\ell(x)$ has two fixed points: -1 and z, where $0 < z < 1$. For each $x \in [-z, z] \setminus \{0\}$, there exists $i_x \in \mathbf{N}$ such that $f_\ell^i(x) \notin [-z, z]$ for $i = 1, \dots, i_x - 1$ and $f_\ell^{i_x}(x) \in [-z, z]$. Define $R_\ell(x) := f_\ell^{i_x}(x)$. The domain of R_ℓ (i.e., $[-z, z] \setminus \{0\}$) can be divided into maximal intervals J_j, $1 \leq j < \infty$, on which the return time (i.e., i_x) is a constant which is denoted by $k(j)$. The following proposition says that R_ℓ has a continuous invariant density λ and for $x \in J$, in addition, $\rho(x)$ is equal to $\lambda(x)$ multiplied by a constant.

Proposition 2.3 (see [17], pp. 363–365). *There exists an R_ℓ-invariant probability measure m which is absolutely continuous with respect to Lebesgue measure. Let $\lambda(x)$ be its density* (*defined only on J*). *Then the following properties hold.*

1. *$\lambda(x)$ is Lipschitz continuous.*
2. *$\lambda(x)$ is uniformly bounded away from zero.*
3. For $x \in J$, *the f_ℓ-invariant density ρ is given by $\rho(x) = \lambda(x)/\Gamma$, where $\Gamma := \sum_{j=1}^{\infty} k(j)m(J_j) < \infty$.*

Proof of Proposition 2.1. From Proposition 2.3, $\rho(x)$ is Lipschitz continuous on J. Let $y > 0$ and $-y$ be the inverse images $f_\ell^{-1}(x)$ for $x \in (-1, 1)$. The Perron–Frobenius operator Φ applied to ρ yields

$$\Phi(\rho)(x) = \frac{\rho(-y)}{|f_\ell'(-y)|} + \frac{\rho(y)}{|f_\ell'(y)|} = \frac{\rho(-y)+\rho(y)}{|f_\ell'(y)|}. \tag{2.5}$$

But $\rho = \Phi(\rho)$ since ρ is invariant. Hence we have

$$\rho(x) = \frac{\rho(-y)+\rho(y)}{|f_\ell'(y)|} = \frac{\rho(-y)+\rho(y)}{2\ell|y|^{\ell-1}}. \tag{2.6}$$

If $x \in [z, 1)$, then $y \in J$. (Recall that $J = [-z, z]$.) Notice that ρ is continuous on J. Thus Eq. (2.6) implies ρ is continuous at z, i.e. we need to show that the limits of ρ at z from both sides, denoted by $\rho(z_-)$ and $\rho(z_+)$, respectively, are equal, or in other words, we need to prove that

$$\rho(z_-) = d[\rho(z_-) + \rho(-z_+)], \tag{2.7}$$

where $d := 1/|f_\ell'(z)|$ and $\rho(-z_+)$ denotes the right limit of ρ at $-z$.

Let $-w_{-1}$ be the unique point in $f^{-1}(-z) \cap (-1, 0)$. For $1 \le i < \infty$, let $-w_{-(i+1)}$ be the unique point in $f^{-1}(-w_{-i}) \cap (-1, 0)$, and let $z_{-(i+1)}$ be the unique point in $f^{-1}(w_{-i}) \cap (-1, 0)$. For $2 \le i < \infty$, it is clear that $z_{-i} \in J$, $f^i(z_{-i}) = -z$, and the return time equals 2 on (z_{-2}, z) and is constant on each $(z_{-(i+1)}, z_{-i})$. Applying Eq. (2.6) repeatedly yields

$$\rho(z_-) = d^2\rho(z_-) + d^2\rho(-z_+) + \sum_{i=2}^{\infty} \frac{\rho(z_{-i})}{|(f_\ell^{i+1})'(z_{-i})|} + \sum_{i=2}^{\infty} \frac{\rho(-z_{-i})}{|(f_\ell^{i+1})'(-z_{-i})|} \tag{2.8}$$

and

$$\rho(-z_+) = \sum_{i=2}^{\infty} \frac{\rho(z_{-i})}{|(f_\ell^{i})'(z_{-i})|} + \sum_{i=2}^{\infty} \frac{\rho(-z_{-i})}{|(f_\ell^{i})'(-z_{-i})|}. \tag{2.9}$$

Since $|(f_\ell^{i+1})'(-z_{-i})| = |(f_\ell^{i+1})'(z_{-i})| = |f'(f_\ell^i(z_{-i})) \cdot (f_\ell^i)'(z_{-i})| = |(f_\ell^i)'(z_{-i})|/d$, Eq. (2.8) can be rewritten as

$$\rho(z_-) = d^2\rho(z_-) + d^2\rho(-z_+) + d\left[\sum_{i=2}^{\infty} \frac{\rho(z_{-i})}{|(f_\ell^{i})'(z_{-i})|} + \sum_{i=2}^{\infty} \frac{\rho(-z_{-i})}{|(f_\ell^{i})'(-z_{-i})|}\right].$$

Substituting (2.9) into the above equation, we get

$$\rho(z_-) = d^2\rho(z_-) + d^2\rho(-z_+) + d\rho(-z_+).$$

Solving for $\rho(z_-)$ yields

$$\rho(z_-) = \frac{d}{1-d}\rho(-z_+),$$

which implies (2.7).

The continuity of ρ on $(-1, -z]$ can be proved as follows. For each $x \in [-w_{-1}, -z]$, Eq. (2.6) yields

$$\rho(f_\ell(x)) = \frac{\rho(x) + \rho(-x)}{|f_\ell'(x)|}. \tag{2.10}$$

We can solve $\rho(x)$ from (2.10) and get

$$\rho(x) = |f_\ell'(x)|\rho(f_\ell(x)) - \rho(-x).$$

Since both $f_\ell(x)$ and $-x$ are contained in $[-z, 1)$, the above equation implies that ρ is continuous at x, and that the limits of ρ at $-z$ from both sides coincide. Therefore ρ is continuous on $[-w_{-1}, 1)$. Repeating previous arguments, and noticing that $\lim_{i\to\infty} - w_{-i} = -1$, we conclude that ρ is continuous on $(-1, 1)$.

In the following we prove that the limits in (2.1) and (2.2) exist and Eq. (2.3) holds. We first notice that Eq. (2.6) implies that the limit in (2.1) exists, since

$$\alpha_1 = \lim_{y\to 0} \frac{\rho(-y) + \rho(y)}{2\ell|y|^{\ell-1}} \cdot (2|y|^\ell)^{1-1/\ell} = \frac{2^{1-1/\ell}\rho(0)}{\ell}.$$

To prove the limit in (2.2) exists it suffices to show that $\rho(x)(1+x)^{1-1/\ell}$ is bounded on $(-1,-z]$, and that the "limsup" and "liminf" coincide.

Given $x \in (-1,-z]$, let $y_{-1} := y$, where as defined earlier $y > 0$ and $-y$ are the inverse images of $f_\ell^{-1}(x)$. For $1 \le i < \infty$, let $y_{-(i+1)}$ be the unique point in $f_\ell^{-1}(-y_{-i}) \cap (0,1)$. Then $f_\ell^i(y_{-i}) = x$. Applying (2.6) repeatedly yields

$$\rho(x) = \sum_{i=1}^{\infty} \frac{\rho(y_{-i})}{|(f_\ell^i)'(y_{-i})|}.$$

Let $D := \sup_{0\le y<1} \rho(y)(1-y)^{1-1/\ell}$. Since ρ is continuous on $[0,1)$ and the limit in (2.1) exists, D is a real number. Also notice that f_ℓ has negative Schwarzian derivative, so it has bounded distortion [17, Theorem IV 1.2] on $(-1,-z]$. Thus there exists a constant $E > 0$ such that

$$|(f_\ell)'(-z)|^i \le \left(\frac{1+x}{1-y_{-i}}\right) = \left(\frac{f_\ell^i(-y_{-i}) - (-1)}{-y_{-i} - (-1)}\right) \le E|(f_\ell^i)'(y_{-i})|. \tag{2.11}$$

Thus

$$\begin{aligned}\rho(x)(1+x)^{1-1/\ell} &= \sum_{i=1}^{\infty} \frac{\rho(y_{-i})}{|(f_\ell^i)'(y_{-i})|}(1+x)^{1-1/\ell} = \sum_{i=1}^{\infty} \frac{\rho(y_{-i})(1-y_{-i})^{1-1/\ell}}{|(f_\ell^i)'(y_{-i})|} \cdot \left(\frac{1+x}{1-y_{-i}}\right)^{1-1/\ell} \\ &\le \sum_{i=1}^{\infty} \frac{D\{E|(f_\ell^i)'(y_{-i})|\}^{1-1/\ell}}{|(f_\ell^i)'(y_{-i})|} \le DE^{1-1/\ell} \sum_{i=1}^{\infty} |f_\ell'(z)|^{-i/\ell} < \frac{DE^{1-1/\ell}}{|(f_\ell^i)(z)|^{1/\ell} - 1} < \infty.\end{aligned}$$

Hence $\rho(x)(1+x)^{1-1/\ell}$ is bounded in $(-1,-z]$.

From Eq. (2.6), we have

$$\begin{aligned}&\limsup_{x\to -1} \rho(x)(1+x)^{1-1/\ell} \\ &\quad = \limsup_{y\to 1} \frac{(\rho(-y)+\rho(y))(2-2y^\ell)^{1-1/\ell}}{2\ell y^{\ell-1}} = \limsup_{y\to 1} \frac{(\rho(-y)+\rho(y))(2\ell)^{1-1/\ell}(1-y)^{1-1/\ell}}{2\ell y^{\ell-1}} \\ &\quad = (2\ell)^{-1/\ell}\left(\limsup_{y\to 1} \rho(-y)(1-y)^{1-1/\ell} + \alpha_1\right) = (2\ell)^{-1/\ell}\left(\limsup_{y\to 1} \rho(x)(1+x)^{1-1/\ell} + \alpha_1\right).\end{aligned}$$

Solving for the "limsup" yields

$$\limsup_{x\to -1} \rho(x)(1+x)^{1-1/\ell} = \frac{\alpha_1}{(2\ell)^{1/\ell} - 1}.$$

Similarly the "liminf" in (2.2) is the same; hence the limit exists, and the relation (2.3) holds. □

Remark. *Keller* [18] (*see also* [17, *Theorem* V 3.2]) *has proved that for a unimodal map with one non-flat critical point* (*i.e., it is of finite order*) *and negative Schwarzian derivative, the existence of an absolutely continuous probability measure implies that Lyapunov exponents are positive for almost all initial points. In this sense,* f_ℓ $(\ell > 2)$ *is chaotic.*

3. Proof of main results

In this section we prove Theorem 1.1 after preliminary lemmas. We apply the methods introduced heuristically by Grebogi et al. [3], but we use them for a rigorous argument to estimate $\mathcal{P}_{collapse}$ quantitatively.

Indeed, $\mathcal{P}_{collapse}$ can be equivalently defined in terms of probabilities of sequences without reference to a space of maps. For a sequene $\sigma = (x_{i_j})_{j=1}^n$ of points in $\Delta_N \setminus \{-1\}$, let x_{i_s} denote the first term (if it exists) for which either x_{i_s} equals an earlier x_{i_j} or $x_{i_s} = 1$. In the latter case, we say σ is *collapsing*, and s is the *collapsing time*; in the first case, we say σ is *repetitive*, and s is the *repeating time*. There are sequences that are neither collapsing nor repetitive, but their length is at most $2N - 1$, since there are only $2N - 1$ different points in $\Delta_N \setminus \{-1, 1\}$. Define $\mathcal{P}'_{collapse}$ as the probability that an (infinite) sequence chosen at random (using the probabilities in (1.2)) is collapsing, that is, $x_{i_j} = 1$ occurs before the first repeat.

The definitions $\mathcal{P}_{collapse}$ and $\mathcal{P}'_{collapse}$ are equivalent. This is because if $(x_{i_j})_{j=1}^n$ is a sequence in $\{x_1, \dots, x_{2N-1}\}$ such that $x_{i_1} \neq \cdots \neq x_{i_n}$, i.e., no pair of indices in $\{i_1, \dots, i_n\}$ are equal, then the probability of choosing this sequence is the same as the probability of obtaining this sequence as a trajectory of a randomly chosen map (using the probabilities in (1.3)). Therefore, a collapsing sequence $(x_{i_j})_{j=1}^s$ with collapsing time s is chosen with a probability same as that with which the orbit $(x_{i_j})_{j=1}^s$ of a map T is chosen from $\mathcal{T}_{N,\ell}$.

We first give a heuristic argument before proving Theorem 1.1 in detail. Let $\sigma = (x_{i_j})_{j=1}^\infty$ be a sequence chosen at random. Since the probability of picking 1 is equal to p_{2N}, $\mathcal{P}_{collapse}$ is roughly equal to p_{2N} times the average length of a maximal sequence that is neither collapsing nor repetitive. The chance of repetition is small for short sequences and gets larger for longer sequences. Define

$$Proc(n) := P\{(x_{i_j})_{j=1}^n \text{ has no 1's and no repeats}\}, \tag{3.1}$$

$$Stop(n) := 1 - Proc(n). \tag{3.2}$$

Then the average length of maximal sequences mentioned earlier is approximately equal to the cufoff of N_{stop} where $Stop(N_{stop})$ is not negligible. Indeed, we will see in Eq. (3.6) that this cutoff is approximately equal to the minimum of p_{2N}^{-1} and $\langle p\rangle^{-1}$ and $\langle p\rangle^{-1/2}$, where $\langle p\rangle := \sum_{j=1}^{2N-1} p_j^2$. $\langle p\rangle$ is the "average correlation" between two points chosen at random from Δ_N and is of the order N^{-D_2}, where D_2 is the *correlation dimension* of f_ℓ defined by $D_2 := \lim_{N\to\infty} - \log\langle p\rangle / \log N$. (This definition is slightly different from the standard definition, e.g., see [19], but it is obvious that they are equivalent.) We will see in Proposition 3.3 that $D = 2/\ell$ and $\langle p\rangle \sim N^{-2/\ell}$. By definition, p_{2N} is the measure at 1. From Proposition 2.1, $p_{2N} \sim N^{-1/\ell}$. Thus $N_{stop} \sim N^{1/\ell}$, hence $\mathcal{P}_{collapse} \sim p_{2N} \cdot N_{stop} = O(1)$.

The following notation is needed to estimate $Stop(n)$. For $1 \le j \le n$, define

$$Prob(n, k, j) := P\{x_{i_j} = x_k | (x_{i_r})_{r=1}^n \text{ is neither collapsing nor repetitive}\}. \tag{3.3}$$

Notice that given two sequences that are neither repetitive nor collapsing if they contain the same set of points and the order of these points is different, they have the same probability to be chosen. Thus $Prob(n, k, j)$ is independent of j and therefore we can suppress the notation by $Prob(n, k)$.

Notice that $Prob(n, k)$ is typically not equal to p_k due to the restriction that the first n points are different from each other. To illustrate how this deviation can happen, we consider a simple example in which a biased coin is tossed at random. Suppose in a single experiment the head occurs with probability 0.9 and the tail occurs with probability 0.1. If we toss this coin twice, then under the condition that both the head and the tail occur once, the probability that the head occurs first is 0.5, which is far from 0.9. On the other hand, when n is small compared to N, choosing n points at random from Δ_N is unlikely to result in repetition, so $Prob(n, k)/p_k$ is close to 1, as shown in the next lemma.

Lemma 3.1. *Let N be sufficiently large that $p_{2N} < 1/32$. Given $\theta \in (2p_{2N}, 1/16)$, let $N_\theta = \theta/\langle p\rangle$, then for all $n \le N_\theta$,*

$$Prob(n,k) < (1+2\theta)p_k. \tag{3.4}$$

Proof. As before, we denote by $i_1 \ne \cdots \ne i_n$ if no pair of indices in $\{i_1,\dots,i_n\}$ are equal, and also we write $k \ne i_1,\dots,i_n$ if k does not belong to the index set $\{i_1,\dots,i_n\}$.

Let $(x_{i_j})_{j=1}^n$ be a sequence that is neither collapsing nor repetitive. Then

$$\begin{aligned}
Prob(n,k) &= P\{i_1=k|i_1\ne\cdots\ne i_n\ne 2N\} = \frac{P\{i_1=k|i_1\ne\cdots\ne i_n\ne 2N\}}{P\{i_1\ne\cdots\ne i_n\ne 2N\}}\\
&= \frac{P\{i_1=k\}P\{i_1\ne\cdots\ne i_n\ne 2N|i_1=k\}}{P\{i_1\ne\cdots\ne i_n\ne 2N\}}\\
&= \frac{p_kP\{i_2\ne\cdots\ne i_n\ne 2N|i_1=k\}P\{k\ne i_2,i_3,\dots,i_n|i_2\ne\cdots\ne i_n\ne 2N\}}{P\{i_2\ne\cdots\ne i_n\ne 2N\}P\{i_1\ne i_2,i_3,\dots,i_n|i_2\ne\cdots\ne i_n\ne 2N\}}\\
&= \frac{p_kP\{k\ne i_2,i_3,\dots,i_n|i_2\ne\cdots\ne i_n\ne 2N\}}{P\{i_1\ne i_2,i_3,\dots,i_n,2N|i_2\ne\cdots\ne i_n\ne 2N\}}\\
&= \frac{p_k - p_k\sum_{s=2}^n P\{i_s=k|i_2\ne\cdots\ne i_n\ne 2N\}}{1-p_{2N}-\sum_{j=1}^{2N-1}[p_j\sum_{s=2}^n P\{i_s=j|i_2\ne\cdots\ne i_n\}]}\\
&= \frac{p_k - p_k(n-1)Prob(n-1,k)}{1-p_{2N}-\sum_{j=1}^{2N-1}[p_j(n-1)Prob(n-1,j)]}.
\end{aligned}$$

In particular, $p(1,k) = p_k/(1-P_{2N}) < (1+2\theta)p_k$.

Assume, for induction, that $Prob(n-1,k) < (1+2\theta)p_k$. Then

$$\begin{aligned}
\frac{Prob(n,k)}{p_k} &\le \left\{1-p_{2N}-\sum_{j=1}^{2N-1}[(n-1)p_j^2(1+2\theta)]\right\}^{-1} = \{1-p_{2N}-(1+2\theta)(n-1)\langle p\rangle\}^{-1}\\
&\le \left\{1-\frac{\theta}{2}-(1+2\theta)\theta\right\}^{-1}.
\end{aligned}$$

It can be easily seen that for $\theta \in (2p_{2N}, 1/16)$, the last expression in the above is less than $1+2\theta$. Thus we complete the proof. □

Let $Rep(n)$ be the probability that $(x_{i_j})_{j=1}^{n+1}$ is repetitive, given that the subsequence consisting of the first n points is neither repetitive nor collapsing. Then

$$Rep(n) = \sum_{k=1}^{2N-1}[n\,Prob(n,k)p_k].$$

From (3.4), $n \le N_\theta$ implies

$$Rep(n) \le n\langle p\rangle(1+2\theta). \tag{3.5}$$

Noticing that $Proc(n) = Proc(n-1)\cdot(1-Rep(n)-p_{2N})$, where $Proc(n)$ is defined in (3.1), we obtain

$$\log Proc(n) = \log([1 - Rep(1) - p_{2N}]\cdots[1 - Rep(n-1) - p_{2N}]) \geq \sum_{i=1}^{n-1} \log(1 - (1+2\theta)\langle p\rangle i - p_{2N}),$$

$$\text{from}(3.5) \geq \sum_{i=1}^{n-1}\{[-i\langle p\rangle(1+2\theta) - p_{2N}] - [i\langle p\rangle(1+2\theta) + p_{2N}]^2\} \geq -(1+2\theta)\langle p\rangle \frac{n^2}{2} - np_{2N}. \quad (3.6)$$

Let

$$Collapse(n) := P\{(x_{i_j})_{j=1}^{\infty} \text{ is collapsing with collapsing time } n+1\}. \quad (3.7)$$

Then from Eq. (3.6), for $n \leq N_\theta$, we have

$$Collapse(n) = p_{2N} Proc(n) \geq p_{2N} \exp\left[-(1+2\theta)\langle p\rangle \frac{n^2}{2} - np_{2N}\right], \quad (3.8)$$

from Eq. (3.6).

We now give results which allow us to estimate p_{2N} and $\langle p\rangle$ asymptotically as $N \to \infty$.

Lemma 3.2.

$$\lim_{N\to\infty} p_{2N} \cdot (2N)^{1/\ell} = \alpha_1 \ell. \quad (3.9)$$

Proof.

$$\lim_{N\to\infty} p_{2N} \cdot (2N)^{1/\ell} = \lim_{N\to\infty} \left(\frac{1}{1-\tilde{p}_0}\right) \int_{1-1/2N}^{1} p(x)\,\mathrm{d}x \cdot (2N)^{1/\ell}$$
$$= \lim_{N\to\infty} \int_{1-1/2N}^{1} \sigma_1 (1-x)^{1/\ell - 1} \mathrm{d}x (2N)^{1/\ell} = \sigma_1 \ell,$$

where the second step follows from Eq. (2.1). □

In the following proposition, the dimension calculation is a consequence of Eq. (3.11), but Eq. (3.11) is also needed in calculating $Collapse(n)$. The following constant is needed:

$$K := \left(\sum_{i=1}^{\infty} k_i^2\right)^{1/2}, \quad (3.10)$$

where

$$k_i := \ell[(i+1/2)^{1/\ell} - (i-1/2)^{1/\ell}] = N^{1/\ell}\left(\int_{x_i - 1/(2N)}^{x_i + 1/(2N)} (1+x)^{-1+1/\ell} \mathrm{d}x\right).$$

Proposition 3.3.

1. $\lim_{N\to\infty} \langle p\rangle N^{2/\ell} = (\alpha_0^2 + \alpha_1^2) K^2.$ (3.11)
2. $D_2 = 2/\ell.$ (3.12)

Proof. From Proposition 2.1, for fixed $\delta > 0$, there exist $a, b \in (-1, 1)$ such that

$$(1-\delta)\alpha_0 (1+x)^{1/\ell - 1} \leq \rho(x) \leq (1+\delta)\alpha_0(1+x)^{1/\ell - 1}, \quad \text{for } x < a, \quad (3.13)$$

$$(1-\delta)\alpha_1 (1-x)^{1/\ell - 1} \leq \rho(x) \leq (1+\delta)\alpha_1(1-x)^{1/\ell - 1}, \quad \text{for } x > b. \quad (3.14)$$

Writing $I_1 := \sum_{-1<x_i<a} \tilde{p}_i^2$, $I_2 := \sum_{a\le x_i\le b} \tilde{p}_i^2$, and $I_3 := \sum_{b<x_i<1} \tilde{p}_i^2$, so that

$$\langle p\rangle = [I_1 + I_2 + I_3] \left(\frac{1}{1-\tilde{p}_0}\right)^2 .$$

Notice that $\lim_{N\to\infty}(1-\tilde{p}^{-2}) = 1$. Therefore we obtain

$$\lim_{N\to\infty} \langle p\rangle N^{2/\ell} = \lim_{N\to\infty} (I_1 + I_2 + I_3) N^{2/\ell}. \tag{3.15}$$

In the following we estimate the quantities I_1, I_2 and I_3 separately. Since

$$\tilde{p}_i^2 = \left(\int_{x_i-1/2N}^{x_i+1/2N} \rho(x)\,\mathrm{d}x\right)^2,$$

Eq. (3.13) yields

$$I_1 \le (1+\delta)^2\alpha_0^2 \sum_{-1<x_i<a} \left[\int_{x_i-1/2N}^{x_i+1/2N} (1+x)^{1/\ell-1}\mathrm{d}x\right]^2 = (1+\delta)^2\alpha_0^2 \sum_{0<i<(1+a)N} k_i^2 N^{-2/\ell},$$

and

$$I_1 \ge (1-\delta)^2\alpha_0^2 \sum_{0<i<(1+a)N} k_i^2 N^{-2/\ell}.$$

Similarly, Eq. (3.14) yields

$$I_3 \le (1+\delta)^2\alpha_1^2 \sum_{0<i<(1-b)N} k_i^2 N^{-2/\ell}.$$

and

$$I_3 \ge (1-\delta)^2\alpha_1^2 \sum_{0<i<(1-b)N} k_i^2 N^{-2/\ell}.$$

Let $M = \max_{a\le x\le b}\rho(x)$. Then $I_2 \le M^2/(2N)$. Notice that $\lim_{N\to\infty} I_2 N^{2/\ell} = 0$. Combined with the above estimates, Eq. (3.15) gives

$$(1-\delta)^2(\alpha_0^2+\alpha_1^2)\sum_{i=1}^{\infty} k_i^2 \le \liminf_{N\to\infty}\langle p\rangle N^{2/\ell} \le \limsup_{N\to\infty}\langle p\rangle N^{2/\ell} \le (1+\delta)^2(\alpha_0^2+\alpha_1^2)\sum_{i=1}^{\infty} k_i^2.$$

Since δ can be chosen to be arbitrarily small, the leftmost expression in the above equals the rightmost expression, so the four expressions are equal and therefore Eq. (3.11) holds. □

Remark. *In* [3], *numerical experiments and heuristic estimates are given to argue that it appears that for chaotic systems the average period is of order $\epsilon^{-D_2/2}$, where ϵ is the "machine epsilon", i.e., the smallest positive number that can be represented by the computer when evaluating a function using fixed precision. In effect, $\epsilon = 1/N$. If the arguments in* [3] *hold for f_ℓ, then that means the average period of a numerical orbit is of order $\epsilon^{-1/\ell}$. Later we will see that numerical experiments suggest that the average collapsing time is also approximately $\epsilon^{-1/\ell}$.*

Proof of Theorem 1.1. For $n > 0$, let $Collapse(n)$ be defined by (3.7). Define

$$\mathcal{P}_N(n) := P\{(x_{i_j})_{i=1}^{n+1} \text{ is collapsing}\}. \tag{3.16}$$

Notice that $\mathcal{P}_{collapse} = \mathcal{P}_N(2N-1)$. Then for $n \le N_\theta$, we have

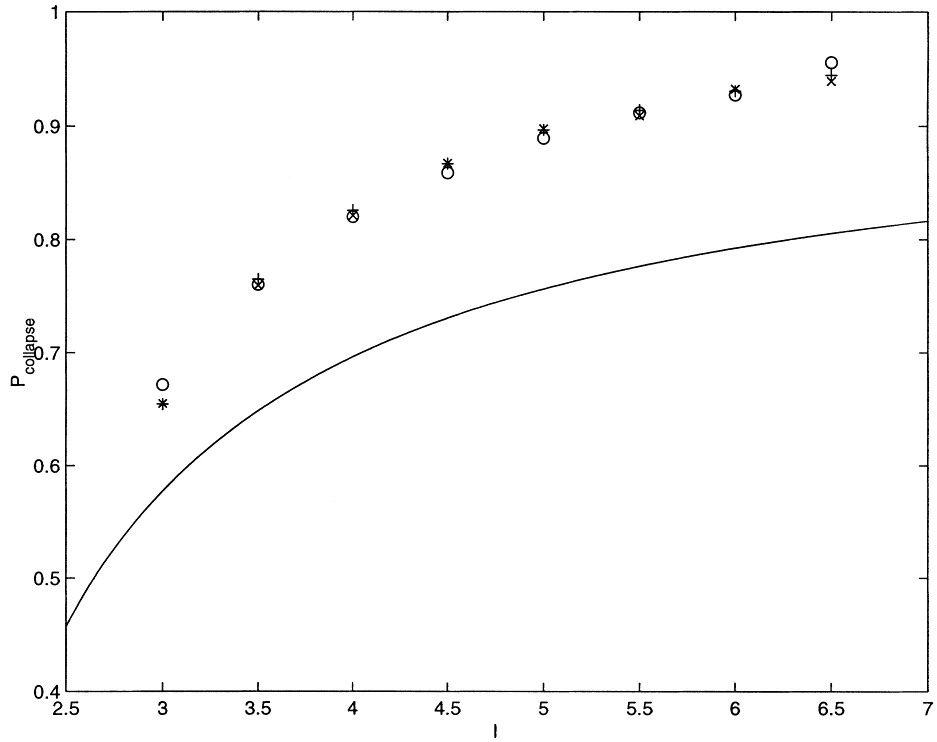

Fig. 1. $\mathcal{P}_{collapse}$ as a function of ℓ. The curve in this figure is the lower bound computed from (3.23). Numerical results obtained by using different numerical precisions are also shown in this figure: '+' – double precision; 'o' – single precision; '×' – fixed precision 10^{-12}.

$$
\begin{aligned}
\mathcal{P}_N(n) &= \sum_{i=0}^{n} Collapse(i) \ge p_{2N} \sum_{i=0}^{n} \exp\left[-(1+2\theta)\langle p\rangle \frac{i^2}{2} - i p_{2N}\right], \quad \text{from Eq. (3.8)},\\
&\ge p_{2N} \int_0^n \exp\left[-(1+2\theta)\langle p\rangle \frac{x^2}{2} - x p_{2N}\right] \mathrm{d}x\\
&\ge \sqrt{\pi} L(\theta) \left[\mathrm{erf}\left(\frac{n p_{2N}}{2L(\theta)} + L(\theta)\right) - \mathrm{erf}(L(\theta))\right] \exp(L(\theta)^2),
\end{aligned} \tag{3.17}
$$

and

$$
\liminf_{N\to\infty} \mathcal{P}_{collapse} \ge \lim_{\theta\to 0} \liminf_{N\to\infty} \sqrt{\pi} L(\theta) \left[\mathrm{erf}\left(\frac{N_\theta p_{2N}}{2L(\theta)} + L(\theta)\right) - \mathrm{erf}(L(\theta))\right] \exp(L(\theta)^2), \tag{3.18}
$$

where $\mathrm{erf}(x) := (2/\sqrt{\pi}) \int_0^x \exp(-t^2)\,\mathrm{d}t$, and

$$
L(\theta) := \frac{p_{2N}}{\sqrt{2(1+2\theta)\langle p\rangle}}. \tag{3.19}
$$

Multiplying the numerator and denominator by $(2N)^{1/\ell}$ and using Eqs. (3.9) and (3.11) yields

$$
\lim_{N\to\infty} L(\theta) = \ell K^{-1} 2^{-1/2-1/\ell} (1+2\theta)^{-1/2} \sqrt{\frac{\alpha_1^2}{\alpha_0^2+\alpha_1^2}}, \tag{3.20}
$$

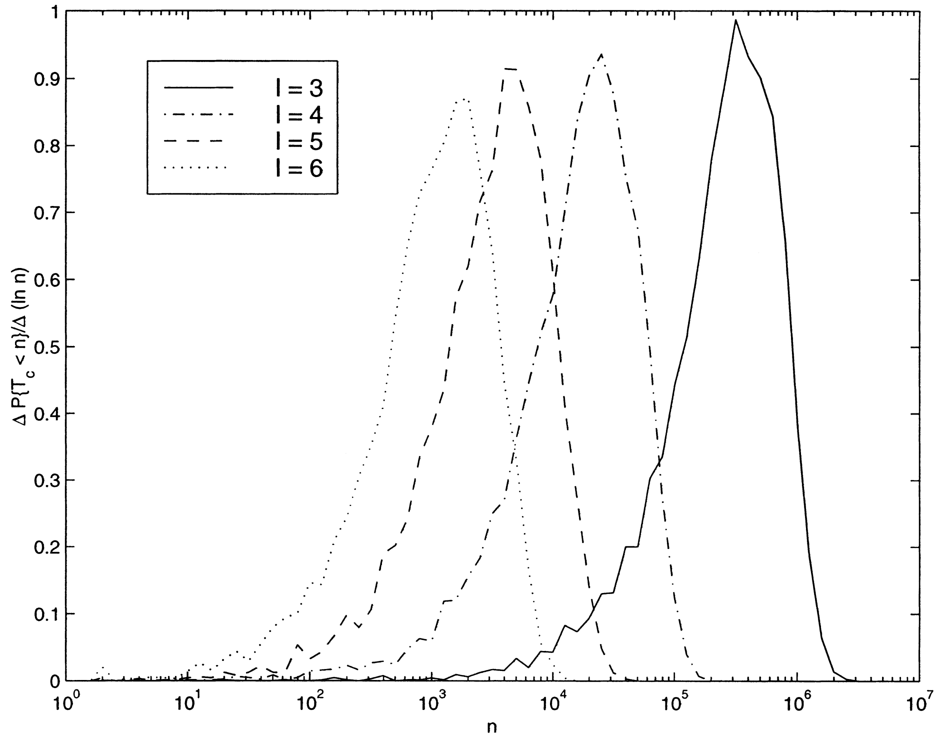

Fig. 2. The distribution of collapsing time T_c for double precision computation, $\ell = 3, 4, 5, 6$. The density is normalized by $\mathcal{P}_{collapse}$.

and by substituting (2.3) into the above equation, we get

$$\lim_{N\to\infty} L(\theta) = \ell K^{-1} 2^{-1/2-1/\ell}(1+2\theta)^{-1/2}[1+((2\ell)^{1/\ell}-1)^{-2}]^{-1/2}, \tag{3.21}$$

where K is given by Eq. (3.10).

Recall that $N_\theta := \theta/\langle p\rangle$, for $\theta \in (2p_{2N}, 1/16)$. Thus Eqs. (3.9) and (3.11) yield $\lim_{N\to\infty} N_\theta p_{2N} = \infty$. Hence

$$\lim_{N\to\infty} \operatorname{erf}\left(\frac{N_\theta p_{2N}}{2L(\theta)} + L(\theta)\right) = 1. \tag{3.22}$$

Substituting Eqs. (3.21) and (3.22) into (3.18), we get

$$\liminf_{N\to\infty} \mathcal{P}_{collapse} \geq \sqrt{\pi} K'[1-\operatorname{erf}(K')]\exp(K')^2 > 0, \tag{3.23}$$

where $K' = \lim_{\theta\to 0}\lim_{N\to\infty} L(\theta)$; so Eq. (3.21) yields

$$K' = \ell K^{-1} 2^{-1/2-1/\ell}[1+((2\ell)^{1/\ell}-1)^{-2}]^{-1/2},$$

where K is given by (3.10). □

4. Conclusion

We have not only proved that the collapsing effect does not vanish when arbitrarily high numerical precision is employed, but also given a lower bound of the probability for which it happens (see Eq. (3.23)). In Fig. 1 we plot the curve given by Eq. (3.23) along with the numerical results. Each numerical datum is obtained as follows. For each ℓ, we sample 10 000 pairs of $(\bar{\ell}, x)$ from $(\ell - 0.01, \ell + 0.01) \times (-1, 1)$ with uniform distribution. For each sample, we keep iterating the map $f_{\bar{\ell}}$ with the initial condition x until the numerical trajectory repeats. Then we calculate the portion of trajectories that eventually map to -1. The deviation is clear since Eq. (3.23) only gives a lower bound. Nonetheless, the theoretical curve reveals the fact that $\mathcal{P}_{collapse}$ is already substantial for $\ell = 3$ and it predicts that $\mathcal{P}_{collapse}$ increases as ℓ increases and that $\lim_{l\to\infty}\mathcal{P}_{collapse} = 1$.

Eq. (3.17) allows us to estimate the average collapsing time $\langle T_c \rangle$ for the collapsing trajetories. Roughly speaking, $T_c \sim 1/\sqrt{\langle p \rangle}$. From Proposition 3.3, The collapsing time is related to the correlation dimension as $T_c \sim \epsilon^{-D_2/2}$, where ϵ is the "machine epsilon", and $D_2 = 2/\ell$. Fig. 2 shows the distribution of T_c. We use double precision in our computations, so $\epsilon \approx 10^{-16}$. The peaks of these distributions agree with our prediction up to an order of magnitude.

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